

The Author and The Book

Professor Wilson has over sixty years of professional experience in Civil, Mechanical and Aerospace Engineering. He was a Professor of Structural Engineering at the University of California at Berkeley during the period 1965 to 1991 and has published over 180 papers and books. His research and development contributions have earned him many awards including the election to the National Academy of Engineering in 1985.

Professor Wilson wrote the first automated finite element analysis computer program in 1961 and was the original developer of the **CAL**, **SAP** and **ETABS** series of computer programs. These programs are noted for their accuracy, speed, use of very efficient numerical algorithms and accurate finite elements. During the past 40 years, Ed Wilson has worked as a Senior Consultant to **CSI** on the programming and documentation of these new methods of computational structural analysis.

The major purpose of this book is to summarize the theoretical development of the finite elements and numerical methods used in the latest versions of the **SAP** and **ETABS** programs. Most of the elements and numerical methods used in these programs are new and are not presented in current textbooks on structural analysis. In addition, the book summarizes the fundamental equations of mechanics.

A minimum mathematical background is required in order to completely understand the material presented in the book. However, an understanding of the physical behavior of real structures is essential. A computer programming background is not required.

A new three-dimensional quadrilateral **SHELL** element, with normal rotational degrees-of-freedom, is presented that is accurate for both thin and thick plates and shells. Therefore, shell elements can be easily connected to classical **FRAME** elements. The three-dimensional **SOLID** element can be used to model both fluids and solids.

Dynamic analysis is presented as a logical extension of static analysis in which inertia and damping forces are added to satisfy equilibrium at every point in time. The use of **Load Dependent Ritz**, **LDR**, vectors in a dynamic analysis produce far more accurate results than if the exact dynamic eigenvectors are used.

The use of **LDR** vectors allows the classical mode superposition method to be extended to nonlinear dynamic analysis by the use of the **Fast Nonlinear Analysis**, **FNA**, method. This new method of nonlinear, dynamic analysis allows structures, with a limited number of nonlinear elements, to be analyzed with almost the same computational time as required for a linear dynamic analysis of the same structure.

This is a **must read** book for all researchers and professionals working in the field of modern structural engineering.

Three Dimensional Static and Dynamic Analysis of Structures

A Physical Approach

With Emphasis on Earthquake Engineering

Edward L. Wilson

Professor Emeritus of Structural Engineering

University of California at Berkeley

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Copies of this publication may be obtained from:

**Computers and Structures, Inc.
1995 University Avenue
Berkeley, California 94704 USA**

**Phone: (510) 845-2177
FAX: (510) 845-4096
e-mail: info@csiberkeley.com**

Preface

This edition of the book contains corrections and additions to the July 1998 edition. Most of the new material that has been added is in response to questions and comments from the users of SAP2000, ETABS and SAFE.

Chapter 22 has been written on the direct use of absolute earthquake displacement loading acting at the base of the structure. Several new types of numerical errors, for absolute displacement loading, are identified. First, the fundamental nature of displacement loading is significantly different from the base acceleration loading traditionally used in earthquake engineering. Second, a smaller integration time step is required to define the earthquake displacement and to solve the dynamic equilibrium equations. Third, a large number of modes are required for absolute displacement loading in order to obtain the same accuracy as produced when base acceleration is used as the loading. Fourth, the 90 percent mass participation rule, intended to assure accuracy of the analysis, does not apply for absolute displacement loading. Finally, the effective modal damping for displacement loading is larger than when acceleration loading is used.

In order to reduce these errors associated with displacement loading a higher order integration method, based on a cubic variation of loads within a time step, is introduced in Chapter 13. In addition, static and dynamic participation factors have been defined which allow the structural engineer to minimize the errors associated with displacement type of loading. In addition, Chapter 19 on viscous damping has been expanded in order to illustrate the physical effects of modal damping on the results of a dynamic analysis.

Appendix H, on the speed of modern personal computers, has been updated. It is now possible to purchase a personal computer for approximately \$1,500 that is 25 times faster than a \$10,000,000 CRAY computer produced in 1974.

Several other additions and modifications have been made in this printing. Please send your comments and questions to ed-wilson1@juno.com.

Edward L. Wilson

April 2000

Personal Remarks

My freshman Physics instructor dogmatically warned the class "do not use an equation you cannot derive". The same instructor once stated that "if a person had five minutes to solve a problem, that their life depended upon, the individual should spend three minutes reading and clearly understanding the problem". For the past forty years these simple, practical remarks have guided my work and I hope that the same philosophy has been passed along to my students. With respect to modern structural engineering, one can restate these remarks as "do not use a structural analysis program unless you fully understand the theory and approximations used within the program" and "do not create a computer model until the loading, material properties and boundary conditions are clearly defined".

Therefore, the major purpose of this book is to present the essential theoretical background in order that the users of computer programs for structural analysis can understand the basic approximations used within the program, verify the results of all analyses and assume professional responsibility for the results. It is assumed that the reader has an understanding of statics, mechanics of solids, and elementary structural analysis. The level of knowledge expected is equal to that of an individual with an undergraduate degree in Civil or Mechanical Engineering. Elementary matrix and vector notations are defined in the Appendices and are used extensively. A background in tensor notation and complex variables is not required.

All equations are developed using a physical approach, since this book is written for the student and professional engineer and not for my academic colleagues. Three dimensional structural analysis is relatively simple due to the high speed of the modern computer. Therefore, all equations are presented in three dimensional form and anisotropic material properties are automatically included. A computer programming background is not necessary in order to use a computer program intelligently. However, detailed numerical algorithms are given in order that the readers completely understand the computational methods that are summarized in this book. The Appendices contain an elementary summary of the numerical methods used; therefore, it should not be necessary to spend additional time reading theoretical research papers in order to understand the theory presented in this book.

The author has developed and published many computational techniques for the static and dynamic analysis of structures. It has been personally satisfying that many members of the engineering profession have found these computational methods useful. Therefore, one reason for compiling this theoretical and application book is to consolidate in one publication this research and development. In addition, the recently developed Fast Nonlinear Analysis (FNA) method and other numerical methods are presented in detail for the first time.

The fundamental physical laws that are the basis of the static and dynamic analysis of structures are over 100 years old. Therefore, anyone who believes they have discovered a new fundamental principle of mechanics is a victim of their own ignorance. This book contains computational tricks that the author has found to be effective for the development of structural analysis programs.

The static and dynamic analysis of structures has been automated to a large degree due to the existence of inexpensive personal computers. However, the field of structural engineering, in my opinion, will never be automated. The idea that an expert-system computer program, with artificial intelligence, will replace a creative human is an insult to all structural engineers.

The material presented in the first edition, *Three Dimensional Dynamic Analysis of Structures*, is included and updated in this book. I am looking forward to additional comments and questions from the readers in order to expand the material in future editions of the book.

Edward L. Wilson
July 1998

TABLE OF CONTENTS

1. Material Properties

- 1.1 Introduction 1-1**
- 1.2 Anisotropic Materials 1-1**
- 1.3 Use of Material Properties within Computer Programs 1-4**
- 1.4 Orthotropic Materials 1-5**
- 1.5 Isotropic Materials 1-5**
- 1.6 Plane Strain Isotropic Materials 1-6**
- 1.7 Plane Stress Isotropic Materials 1-7**
- 1.8 Properties of Fluid-Like Materials 1-8**
- 1.9 Shear and Compression Wave Velocities 1-9**
- 1.1 Axisymmetric Material Properties 1-10**
- 1.11 Force-Deformation Relationships 1-11**
- 1.12 Summary 1-12**
- 1.13 References 1-12**

2. Equilibrium and Compatibility

- 2.1 Introduction 2-1**
- 2.2 Fundamental Equilibrium Equations 2-2**
- 2.3 Stress Resultants - Forces And Moments 2-2**
- 2.4 Compatibility Requirements 2-3**
- 2.5 Strain Displacement Equations 2-4**

- 2.6 Definition of Rotation 2-4**
- 2.7 Equations at Material Interfaces 2-5**
- 2.8 Interface Equations in Finite Element Systems 2-7**
- 2.9 Statically Determinate Structures 2-7**
- 2.1 Displacement Transformation Matrix 2-9**
- 2.11 Element Stiffness and Flexibility Matrices 2-11**
- 2.12 Solution of Statically Determinate System 2-11**
- 2.13 General Solution of Structural Systems 2-12**
- 2.14 Summary 2-13**
- 2.15 References 2-14**

3. *Energy and Work*

- 3.1 Introduction 3-1**
- 3.2 Virtual and Real Work 3-2**
- 3.3 Potential Energy and Kinetic Energy 3-4**
- 3.4 Strain Energy 3-6**
- 3.5 External Work 3-7**
- 3.6 Stationary Energy Principle 3-9**
- 3.7 The Force Method 3-10**
- 3.8 Lagrange's Equation of Motion 3-12**
- 3.9 Conservation of Momentum 3-13**
- 3.1 Summary 3-15**
- 3.11 References 3-16**

4. *One-Dimensional Elements*

- 4.1 Introduction 4-1**
- 4.2 Analysis of an Axial Element 4-2**
- 4.3 Two-Dimensional Frame Element 4-4**
- 4.4 Three-Dimensional Frame Element 4-8**
- 4.5 Member End-Releases 4-12**
- 4.6 Summary 4-13**

5. *Isoparametric Elements*

- 5.1 Introduction 5-1**
- 5.2 A Simple One-Dimensional Example 5-2**
- 5.3 One-Dimensional Integration Formulas 5-4**
- 5.4 Restriction on Locations of Mid-Side Nodes 5-6**
- 5.5 Two-Dimensional Shape Functions 5-6**
- 5.6 Numerical Integration in Two Dimensions 5-10**
- 5.7 Three-Dimensional Shape Functions 5-12**
- 5.8 Triangular and Tetrahedral Elements 5-14**
- 5.9 Summary 5-15**
- 5.1 References 5-16**

6. **Incompatible Elements**

- 6.1 Introduction 6-1**
- 6.2 Elements With Shear Locking 6-2**
- 6.3 Addition of Incompatible Modes 6-3**

6.4	Formation of Element Stiffness Matrix	6-4
6.5	Incompatible Two-Dimensional Elements	6-5
6.6	Example Using Incompatible Displacements	6-6
6.7	Three-Dimensional Incompatible Elements	6-7
6.8	Summary	6-8
6.9	References	6-9
7.	Boundary Conditions and General Constraints	
7.1	Introduction	7-1
7.2	Displacement Boundary Conditions	7-2
7.3	Numerical Problems in Structural Analysis	7-3
7.4	General Theory Associated With Constraints	7-4
7.5	Floor Diaphragm Constraints	7-6
7.6	Rigid Constraints	7-11
7.7	Use of Constraints in Beam-Shell Analysis	7-12
7.8	Use of Constraints in Shear Wall Analysis	7-13
7.9	Use of Constraints for Mesh Transitions	7-14
7.1	Lagrange Multipliers and Penalty Functions	7-16
7.11	Summary	7-17
8.	Plate Bending Elements	
8.1	Introduction	8-1
8.2	The Quadrilateral Element	8-3
8.3	Strain-Displacement Equations	8-7

- 8.4 The Quadrilateral Element Stiffness 8-8**
- 8.5 Satisfying the Patch Test 8-9**
- 8.6 Static Condensation 8-10**
- 8.7 Triangular Plate Bending Element 8-10**
- 8.8 Other Plate Bending Elements 8-10**
- 8.9 Numerical Examples 8-11**
 - 8.9.1 One Element Beam 8-12**
 - 8.9.2 Point Load on Simply Supported Square Plate 8-13**
 - 8.9.3 Uniform Load on Simply Supported Square Plate 8-14**
 - 8.9.4 Evaluation of Triangular Plate Bending Elements 8-15**
 - 8.9.5 Use of Plate Element to Model Torsion in Beams 8-16**
- 8.1 Summary 8-17**
- 8.11 References 8-17**

9. Membrane Element with Normal Rotations

- 9.1 Introduction 9-1**
- 9.2 Basic Assumptions 9-2**
- 9.3 Displacement Approximation 9-3**
- 9.4 Introduction of Node Rotation 9-4**
- 9.5 Strain-Displacement Equations 9-5**
- 9.6 Stress-Strain Relationship 9-6**
- 9.7 Transform Relative to Absolute Rotations 9-6**
- 9.8 Triangular Membrane Element 9-8**

9.9 Numerical Example 9-8

9.1 Summary 9-9

9.11 References 9-10

10. Shell Elements

10.1 Introduction 10-1

10.2 A Simple Quadrilateral Shell Element 10-2

10.3 Modeling Curved Shells with Flat Elements 10-3

10.4 Triangular Shell Elements 10-4

10.5 Use of Solid Elements for Shell Analysis 10-5

10.6 Analysis of The Scordelis-Lo Barrel Vault 10-5

10.7 Hemispherical Shell Example 10-7

10.8 Summary 10-8

10.9 References 10-8

11. Geometric Stiffness and P-Delta Effects

11.1 Definition of Geometric Stiffness 11-1

11.2 Approximate Buckling Analysis 11-3

11.3 P-Delta Analysis of Buildings 11-5

11.4 Equations for Three-Dimensional Buildings 11-8

11.5 The Magnitude of P-Delta Effects 11-9

11.6 P-Delta Analysis without Computer Program Modification 11-10

11.7 Effective Length - K Factors 11-11

11.8 General Formulation of Geometry Stiffness 11-11

11.9 Summary 11-13

11.1 References 11-14

12. Dynamic Analysis

12.1 Introduction 12-1

12.2 Dynamic Equilibrium 12-2

12.3 Step-By-Step Solution Method 12-4

12.4 Mode Superposition Method 12-5

12.5 Response Spectra Analysis 12-5

12.6 Solution in the Frequency Domain 12-6

12.7 Solution of Linear Equations 12-7

12.8 Undamped Harmonic Response 12-7

12.9 Undamped Free Vibrations 12-8

12.1 Summary 12-9

12.11 References 12-10

13. Dynamic Analysis Using Mode Superposition

13.1 Equations to be Solved 13-1

13.2 Transformation to Modal Equations 13-2

13.3 Response Due to Initial Conditions Only 13-4

13.4 General Solution Due to Arbitrary Loading 13-5

13.5 Solution for Periodic Loading 13-10

13.6 Participating Mass Ratios 13-11

13.7 Static Load Participation Ratios 13-13

13.8 Dynamic Load Participation Ratios 13-14

13.9 Summary 13-16

14. Calculation of Stiffness and Mass Orthogonal Vectors

14.1 Introduction 14-1

14.2 Determinate Search Method 14-2

14.3 Sturm Sequence Check 14-3

14.4 Inverse Iteration 14-3

14.5 Gram-Schmidt Orthogonalization 14-4

14.6 Block Subspace Iteration 14-5

14.7 Solution of Singular Systems 14-6

14.8 Generation of Load-Dependent Ritz Vectors 14-7

14.9 A Physical Explanation of the LDR Algorithm 14-9

14.1 Comparison of Solutions Using Eigen And Ritz Vectors 14-11

14.11 Correction for Higher Mode Truncation 14-13

14.12 Vertical Direction Seismic Response 14-15

14.13 Summary 14-18

14.14 References 14-19

15. Dynamic Analysis Using Response Spectrum Seismic Loading

15.1 Introduction 15-1

15.2 Definition of a Response Spectrum 15-2

15.3 Calculation of Modal Response 15-4

- 15.4 Typical Response Spectrum Curves 15-4**
- 15.5 The CQC Method of Modal Combination 15-8**
- 15.6 Numerical Example of Modal Combination 15-9**
- 15.7 Design Spectra 15-12**
- 15.8 Orthogonal Effects in Spectral Analysis 15-13**
 - 15.8.1 Basic Equations for Calculation of Spectral Forces 15-14**
 - 15.8.2 The General CQC3 Method 15-16**
 - 15.8.3 Examples of Three-Dimensional Spectra Analyses 15-17**
 - 15.8.4 Recommendations on Orthogonal Effects 15-21**
- 15.9 Limitations of the Response Spectrum Method 15-21**
 - 15.9.1 Story Drift Calculations 15-21**
 - 15.9.2 Estimation of Spectra Stresses in Beams 15-22**
 - 15.9.3 Design Checks for Steel and Concrete Beams 15-22**
 - 15.9.4 Calculation of Shear Force in Bolts 15-23**
- 15.1 Summary 15-23**
- 15.11 References 15-24**

16. Soil Structure Interaction

- 16.1 Introduction 16-1**
- 16.2 Site Response Analysis 16-2**
- 16.3 Kinematic or Soil Structure Interaction 16-2**
- 16.4 Response Due to Multi-Support Input Motions 16-6**
- 16.5 Analysis of Gravity Dam and Foundation 16-9**

- 16.6 The Massless Foundation Approximation 16-11**
- 16.7 Approximate Radiation Boundary Conditions 16-11**
- 16.8 Use of Springs at the Base of a Structure 16-14**
- 16.9 Summary 16-15**
- 16.1 References 16-15**

17. Seismic Analysis Modeling to Satisfy Building Codes

- 17.1 Introduction 17-1**
- 17.2 Three-Dimensional Computer Model 17-3**
- 17.3 Three-Dimensional Mode Shapes and Frequencies 17-4**
- 17.4 Three-Dimensional Dynamic Analysis 17-8**
 - 17.4.1 Dynamic Design Base Shear 17-9**
 - 17.4.2 Definition of Principal Directions 17-10**
 - 17.4.3 Directional and Orthogonal Effects 17-10**
 - 17.4.4 Basic Method of Seismic Analysis 17-11**
 - 17.4.5 Scaling of Results 17-11**
 - 17.4.6 Dynamic Displacements and Member Forces 17-11**
 - 17.4.7 Torsional Effects 17-12**
- 17.5 Numerical Example 17-12**
- 17.6 Dynamic Analysis Method Summary 17-15**
- 17.7 Summary 17-16**
- 17.8 References 17-18**

18. Fast Nonlinear Analysis

- 18.1 Introduction 18-1**
- 18.2 Structures with a Limited Number of Nonlinear Elements 18-2**
- 18.3 Fundamental Equilibrium Equations 18-3**
- 18.4 Calculation of Nonlinear Forces 18-4**
- 18.5 Transformation to Modal Coordinates 18-5**
- 18.6 Solution of Nonlinear Modal Equations 18-7**
- 18.7 Static Nonlinear Analysis of Frame Structure 18-9**
- 18.8 Dynamic Nonlinear Analysis of Frame Structure 18-12**
- 18.9 Seismic Analysis of Elevated Water Tank 18-14**
- 18.1 Summary 18-15**

19. Linear Viscous Damping

- 19.1 Introduction 19-1**
- 19.2 Energy Dissipation in Real Structures 19-2**
- 19.3 Physical Interpretation of Viscous Damping 19-4**
- 19.4 Modal Damping Violates Dynamic Equilibrium 19-4**
- 19.5 Numerical Example 19-5**
- 19.6 Stiffness and Mass Proportional Damping 19-6**
- 19.7 Calculation of Orthogonal Damping Matrices 19-7**
- 19.8 Structures with Non-Classical Damping 19-9**
- 19.9 Nonlinear Energy Dissipation 19-9**
- 19.1 Summary 19-10**

19.11 References 19-10

20. Dynamic Analysis Using Numerical Integration

20.1 Introduction 20-1

20.2 Newmark Family of Methods 20-2

20.3 Stability of Newmark's Method 20-4

20.4 The Average Acceleration Method 20-5

20.5 Wilson's Factor 20-6

20.6 The Use of Stiffness Proportional Damping 20-7

20.7 The Hilber, Hughes and Taylor Method 20-8

20.8 Selection of a Direct Integration Method 20-9

20.9 Nonlinear Analysis 20-9

20.1 Summary 20-10

20.11 References 20-10

21. Nonlinear Elements

21.1 Introduction 21-1

21.2 General Three-Dimensional Two-Node Element 21-2

21.3 General Plasticity Element 21-3

21.4 Different Positive and Negative Properties 21-5

21.5 The Bilinear Tension-Gap-Yield Element 21-6

21.6 Nonlinear Gap-Crush Element 21-7

21.7 Viscous Damping Elements 21-8

21.8 Three-Dimensional Friction-Gap Element 21-10

21.9 Summary 21-12

22. Seismic Analysis Using Displacement Loading

22.1 Introduction 22-1

22.2 Equilibrium Equations for Displacement Input 22-3

22.3 Use of Pseudo-Static Displacements 22-5

22.4 Solution of Dynamic Equilibrium Equations 22-6

22.5 Numerical Example 22-7

22.5.1 Example Structure 22-7

22.5.2 Earthquake Loading 22-9

22.5.3 Effect of Time Step Size for Zero Damping 22-9

22.5.4 Earthquake Analysis with Finite Damping 22-12

22.5.5 The Effect of Mode Truncation 22-15

22.6 Use of Load Dependent Ritz Vectors 22-17

22.7 Solution Using Step-By-Step Integration 22-18

22.8 Summary 22-20

Appendix A Vector Notation

A.1 Introduction A-1

A.2 Vector Cross Product A-2

A.3 Vectors to Define a Local Reference System A-4

A.4 Fortran Subroutines for Vector Operations A-5

Appendix B Matrix Notation

B.1 Introduction B-1

- B.2 Definition of Matrix Notation B-2**
- B.3 Matrix Transpose and Scalar Multiplication B-4**
- B.4 Definition of a Numerical Operation B-6**
- B.5 Programming Matrix Multiplication B-6**
- B.6 Order of Matrix Multiplication B-7**
- B.7 Summary B-7**

Appendix C Solution or Inversion of Linear Equations

- C.1 Introduction C-1**
- C.2 Numerical Example C-2**
- C.3 The Gauss Elimination Algorithm C-3**
- C.4 Solution of a General Set of Linear Equations C-6**
- C.5 Alternative to Pivoting C-6**
- C.6 Matrix Inversion C-9**
- C.7 Physical Interpretation of Matrix Inversion C-11**
- C.8 Partial Gauss Elimination, Static Condensation and Substructure Analysis C-13**
- C.9 Equations Stored in Banded or Profile Form C-15**
- C.10 LDL Factorization C-16**
- C10.1 Triangularization or Factorization of the A Matrix C-17**
- C10.2 Forward Reduction of the b Matrix C-18**
- C10.3 Calculation of x by Backsubstitution C-19**
- C.11 Diagonal Cancellation and Numerical Accuracy C-20**
- C.12 Summary C-20**

C.13 References C-21

Appendix D The Eigenvalue Problem

D.1 Introduction D-1

D.2 The Jacobi Method D-2

D.3 Calculation of 3d Principal Stresses D-4

D.4 Solution of the General Eigenvalue Problem D-5

D.5 Summary D-6

Appendix E Transformation of Material Properties

E.1 Introduction E-1

E.2 Summary E-4

Appendix F A Displacement-Based Beam Element With Shear Deformations

F.1 Introduction F-1

F.2 Basic Assumptions F-2

F.3 Effective Shear Area F-5

Appendix G Numerical Integration

G.1 Introduction G-1

G.2 One-Dimensional Gauss Quadrature G-2

G.3 Numerical Integration in Two Dimensions G-4

G.4 An Eight-Point Two-Dimensional Rule G-5

G.5 An Eight-Point Lower Order Rule G-6

G.6 A Five-Point Integration Rule G-7

G.7 Three-Dimensional Integration Rules G-8

G.8 Selective Integration G-11

G.9 Summary G-11

Appendix H Speed of Computer Systems

H.1 Introduction H-1

H.2 Definition of One Numerical Operation H-1

H.3 Speed of Different Computer Systems H-2

H.4 Speed of Personal Computer Systems H-3

H.5 Paging Operating Systems H-3

H.6 Summary H-4

Appendix I Method of Least Square

I.1 Simple Example I-1

I.2 General Formulation I-3

I.3 Calculation Of Stresses Within Finite Elements I-4

Appendix J Consistent Earthquake Acceleration and Displacement Records

J.1 Introduction J-1

J.2 Ground Acceleration Records J-2

J.3 Calculation of Acceleration Record From Displacement Record J-3

J.4 Creating Consistent Acceleration Record J-5

J.5 Summary J-8