8.

PLATE BENDING ELEMENTS

Plate Bending is a Simple Extension of Beam Theory

8.1 INTRODUCTION

{ XE "Plate Bending Elements" }Before 1960, plates and slabs were modeled using a grid of beam elements for many civil engineering structures. Only a small number of "closed form" solutions existed for plates of simple geometry and isotropic materials. Even at the present time many slab designs are based on grid models. This classical approximate approach, in general, produces conservative results because it satisfies statics and violates compatibility. However, the internal moment and shear distribution may be incorrect. The use of a converged finite element solution will produce a more consistent design. The fundamental difference between a grid of beam elements and a plate-bending finite element solution is that a twisting moment exists in the finite element model; whereas, the grid model can only produce one-dimensional torsional moments and will not converge to the theoretical solution as the mesh is refined.

{ XE "Plate Bending Elements:Thin Plates" }The following approximations are used to reduce the three-dimensional theory of elasticity to govern the behavior of ***thin plates and beams***:

1. { XE "Plate Bending Elements:Reference Surface" }It is assumed that a *line normal to the reference surface* (neutral axis) of the plate (beam) *remains straight* in the loaded position. This displacement constraint is the same as stating that the in-plane strains are a linear function in the thickness direction. This assumption does not require that the rotation of the normal line to be equal to the rotation of the reference surface; hence, transverse shear deformations are possible.
2. In addition, the normal stress in the thickness direction, which is normally very small compared to the bending stresses, is assumed to be zero for both beams and plates. This is accomplished by using ***plane stress material properties*** in-plane as defined in Chapter 1. Note

that this approximation allows Poisson's ratio strains to exist in the thickness direction.

3. { XE "Kirchhoff Approximation" }If the transverse shearing strains are assumed to be zero, an additional displacement constraint is introduced that states that ***lines normal to the reference surface remain normal to the reference surface after loading***. This approximation is attributed to Kirchhoff and bears his name.

{ XE "Plate Bending Elements:Shearing Deformations" }{ XE "Shearing Deformations" }Classical thin plate theory is based on all three approximations and leads to the development of a fourth order partial differential equation in terms of the normal displacement of the plate. This approach is only possible for plates of constant thickness. Many books and papers, using complicated mathematics, have been written based on this approach. However, the Kirchhoff approximation is not required to develop plate bending finite elements that are accurate, robust and easy to program. At the present time, it is possible to include transverse shearing deformations for thick plates without a loss of accuracy for thin plates.

In this chapter, plate bending theory is presented as an extension of beam theory (see Appendix F) and the equations of three-dimensional elasticity. Hence, no previous background in plate theory is required by the engineer to fully understand the approximations used. Several hundred plate-bending finite elements have been proposed during the past 30 years. However, only one element will be presented here. The element is a three-node triangle or a four-node quadrilateral and is formulated with and without transverse shearing deformations. The formulation is restricted to small displacements and elastic materials. Numerical examples are presented to illustrate the accuracy of the element. The theory presented here is an expanded version of the plate bending element first presented in reference [1] using a variational formulation.

8.2 THE QUADRILATERAL ELEMENT

{ XE "Quadrilateral Element" }First, the formulation for the quadrilateral element will be considered. The same approach applies to the triangular element. A quadrilateral of arbitrary geometry, in a local x-y plane, is shown in Figure 8.1. Note that the parent four-node element, Figure 8.1a, has 16 rotations at the four node points and at the mid-point of each side. The mid-side rotations are then rotated to be normal and tangential to each side. The tangential rotations are then set to zero, reducing the number of degrees-of-freedom to 12, Figure 8.1b. The sides of the element are constrained to be a cubic function in u_z and four displacements are introduced at the corner nodes of the element, Figure 8.1c. Finally, the mid-side rotations are eliminated by static condensation, Figure 8.1d, and a 12 DOF element is produced.

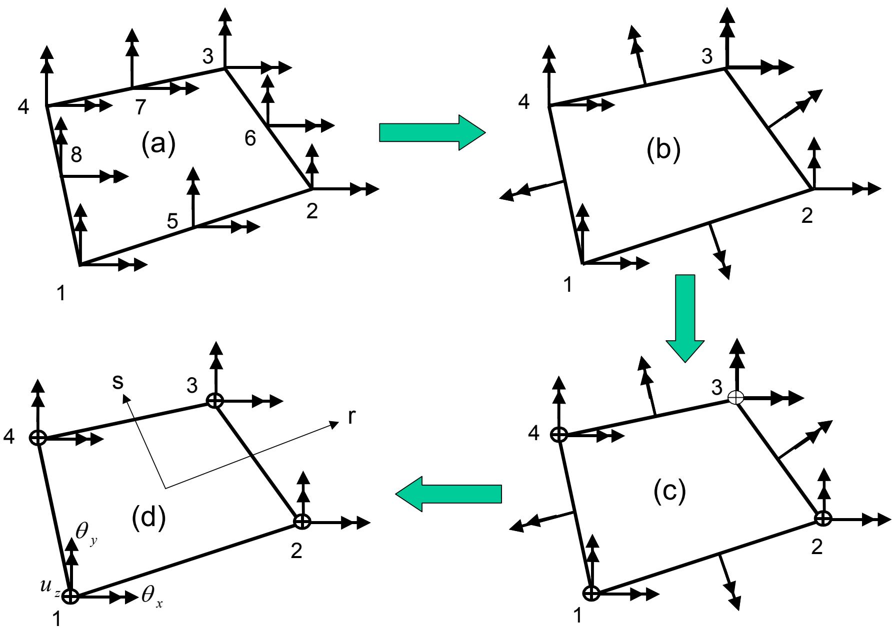

Figure 8.1 Quadrilateral Plate Bending Element

The basic displacement assumption is that the rotation of lines normal to the reference plane of the plate is defined by the following equations:

$$\begin{aligned}\theta_x(r,s) &= \sum_{i=1}^{4} N_i(r,s)\theta_{xi} + \sum_{i=5}^{8} N_i(r,s)\,\Delta\theta_{xi} \\ \theta_y(r,s) &= \sum_{i=1}^{8} N_i(r,s)\theta_{yi} + \sum_{i=5}^{8} N_i(r,s)\,\Delta\theta_{yi}\end{aligned} \tag{8.1}$$

The eight-node shape functions are given by:

$$\begin{aligned} N_1 &= (1-r)(1-s)/\,4 & N_2 &= (1+r)(1-s)/\,4 \\ N_3 &= (1+r)(1+s)/\,4 & N_4 &= (1-r)(1+s)/\,4 \\ N_5 &= (1-r^2)(1-s)/\,2 & N_6 &= (1+r)(1-s^2)/\,2 \\ N_7 &= (1-r^2)(1+s)/\,2 & N_8 &= (1-r)(1-s^2)/\,2 \end{aligned} \tag{8.2}$$

{ XE "Hierarchical Functions" }Note that the first four shape functions are the natural bilinear shape functions for a four-node quadrilateral. The four shape functions for the mid-side nodes are an addition to the bilinear functions and are often referred to as ***hierarchical*** functions. A typical element side *ij* is shown in Figure 8.2.

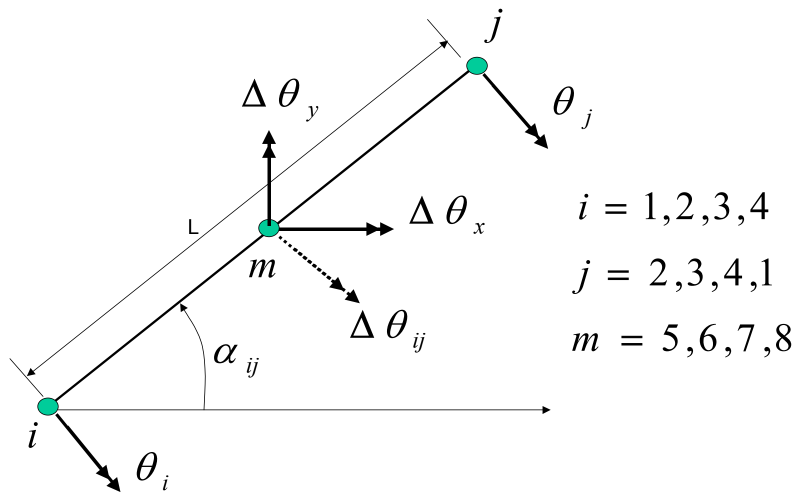

Figure 8.2 Typical Element Side

The tangential rotations are set to zero and only the normal rotations exist. Therefore, the *x* and *y* components of the normal rotation are given by:

$$\begin{aligned} \Delta\theta_x &= \sin\alpha_{ij} \;\; \Delta\theta_{ij} \\ \Delta\theta_y &= -\cos\alpha_{ij} \;\; \Delta\theta_{ij} \end{aligned} \tag{8.3}$$

Hence, Equation (8.1) can be rewritten as:

$$\begin{aligned} \theta_x(r,s) &= \sum_{i=1}^{4} N_i(r,s)\theta_{xi} + \sum_{i=5}^{8} M_{xi}(r,s)\;\Delta\theta_i \\ \theta_y(r,s) &= \sum_{i=1}^{8} N_i(r,s)\theta_{yi} + \sum_{i=5}^{8} M_{yi}(r,s)\;\Delta\theta_i \end{aligned} \tag{8.4}$$

{ XE "Plate Bending Elements:Positive Displacements" }The number of displacement degrees-of-freedom has now been reduced from 16 to 12, as indicated in Figure 8.1b. The three-dimensional displacements, as defined in Figure 8.3 with respect to the x-y reference plane, are:

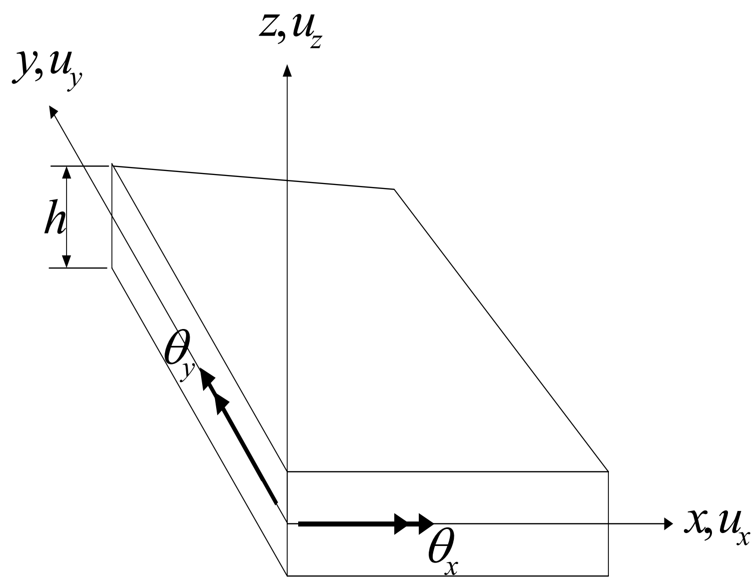

Figure 8.3 Positive Displacements in Plate Bending Element

$$\begin{aligned} u_x(r,s) &= z\,\theta_y(r,s) \\ u_y(r,s) &= -z\,\theta_x(r,s) \end{aligned} \tag{8.5}$$

Note that the normal displacement of the reference plane $u_z(r,s)$ has not been defined as a function of space. Now, it is assumed that the normal displacement along each side is a cubic function. From Appendix F, the transverse shear strain along the side is given by:

$$\gamma_{ij} = \frac{1}{L}(u_{zj} - u_{zi}) - \frac{1}{2}(\theta_i + \theta_j) - \frac{2}{3}\Delta\theta_{ij} \tag{8.6}$$

From Figure 8.2, the normal rotations at nodes *i* and *j* are expressed in terms of the x and y rotations. Or, Equation (8.6) can be written as:

$$\gamma_{ij} = \frac{1}{L}(u_{zj} - u_{zi}) - \frac{\sin\alpha_{ij}}{2}(\theta_{xi} + \theta_{xj}) + \frac{\cos\alpha_{ij}}{2}(\theta_{yi} + \theta_{yj}) - \frac{2}{3}\Delta\theta_{ij} \tag{8.7}$$

This equation can be written for all four sides of the element.

It is now possible to express the node shears in terms of the side shears. A typical node is shown in Figure 8.4.

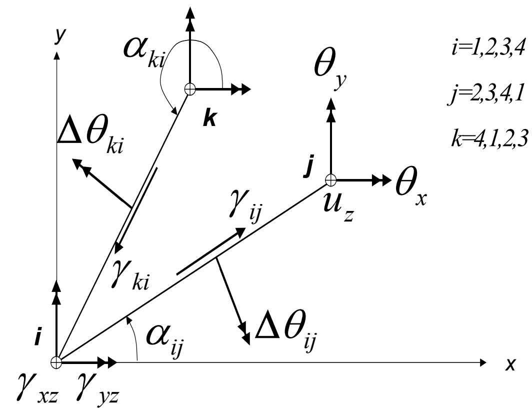

Figure 8.4 Node Point Transverse Shears

The two mid-side shears are related to the shears at node *i* by the following strain transformation:

$$\begin{bmatrix} \gamma_{ij} \\ \gamma_{ki} \end{bmatrix} = \begin{bmatrix} \cos\alpha_{ij} & \sin\alpha_{ij} \\ \cos\alpha_{ki} & \sin\alpha_{ki} \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}_i \tag{8.8}$$

Or, in inverse form:

$$\begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}_i = \frac{1}{\det} \begin{bmatrix} \sin\alpha_{ki} & -\cos\alpha_{ki} \\ -\sin\alpha_{ij} & \cos\alpha_{ij} \end{bmatrix} \begin{bmatrix} \gamma_{ij} \\ \gamma_{ki} \end{bmatrix} \tag{8.9}$$

where $\det = \cos\alpha_{ij} \sin\alpha_{ki} - \cos\alpha_{ki} \sin\alpha_{ij}$.

The final step in determining the transverse shears is to use the standard four-node bilinear functions to evaluate the shears at the integration point.

8.3 STRAIN-DISPLACEMENT EQUATIONS

{ XE "Plate Bending Elements:Strain-Displacement Equations" }{ XE "Strain Displacement Equations:Plate Bending" }Using the three-dimensional strain-displacement equations, the strains within the plate can be expressed in terms of the node rotations. Or:

$$\begin{aligned}\varepsilon_x &= \frac{\partial u_x}{\partial x} = \quad z\,\theta_y(r,s),_x \\ \varepsilon_y &= \frac{\partial u_y}{\partial y} = -z\,\theta_x(r,s),_y \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = z\,[\theta_y(r,s),_y - \theta_x(r,s)]\end{aligned} \tag{8.10}$$

Therefore, at each integration point the five components of strain can be expressed in terms of the 16 displacements, shown in Figure 8.2c, by an equation of the following form:

$$\begin{bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz}\end{bmatrix} = \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{bmatrix} \mathbf{b} \begin{bmatrix}\theta_X \\ \theta_Y \\ u_Z \\ \Delta\theta\end{bmatrix} \text{ or } \mathbf{d} = \mathbf{B}\mathbf{u} = \mathbf{a}(z)\,\mathbf{b}(r,s)\mathbf{u} \tag{8.11}$$

Hence, the strain-displacement transformation matrix is a product of two matrices in which one is a function of z only.

8.4 THE QUADRILATERAL ELEMENT STIFFNESS

{ XE "Plate Bending Elements:Properties" }From Equation (8.11), the element stiffness matrix can be written as:

$$\mathbf{k} = \int \mathbf{B}^{\mathbf{T}}\mathbf{E}\mathbf{B}\,dV = \int \mathbf{b}^T D \mathbf{b}\,dA \tag{8.12}$$

where

$$D = \int \mathbf{a}^T \mathbf{E}\,\mathbf{a}\,dz \tag{8.13}$$

After integration in the z-direction, the 5 by 5 force-deformation relationship for orthotropic materials is of the following form:

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \\ V_{xz} \\ V_{yz}\end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} \\ D_{31} & D_{31} & D_{33} & D_{34} & D_{35} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55}\end{bmatrix} \begin{bmatrix}\psi_{xx} \\ \psi_{yy} \\ \psi_{xy} \\ \gamma_{xz} \\ \gamma_{yz}\end{bmatrix} \tag{8.14}$$

The moments M and shears resultant V are forces per unit length. As in the case of beam elements, the deformations associated with the moment are the curvature ψ. For isotropic plane stress materials, the non-zero terms are given by:

$$D_{11} = D_{22} = \frac{Eh^3}{12(1-\nu^2)}$$

$$D_{12} = D_{21} = \frac{\nu Eh^3}{12(1-\nu^2)} \tag{8.15}$$

$$D_{44} = D_{55} = \frac{5Eh}{12(1+\nu)}$$

8.5 SATISFYING THE PATCH TEST

{ XE "Patch Test" }{ XE "Plate Bending Elements:Patch Test" }For the element to satisfy the patch test, it is necessary that constant curvatures be produced if the node displacements associated with constant curvature are applied. Equation (8.11) can be written in the following form:

$$\begin{bmatrix} \psi_{xx} \\ \psi_{yy} \\ \psi_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{b}_{22} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ w \\ \Delta\theta \end{bmatrix} \tag{8.16}$$

{ XE "Plate Bending Elements:Constant Moment" }where, for a quadrilateral element, $\mathbf{b}_{11}$ is a 3 by 12 matrix associated with the 12 node displacements (θ_x, θ_y, w) and $\mathbf{b}_{12}$ is a 3 by 4 matrix associated with the incompatible 4 normal side rotations ($\Delta\theta$). In order that the element satisfies the constant moment patch test, the following modification to $\mathbf{b}_{12}$ must be made:

$$\overline{\mathbf{b}}_{12} = \mathbf{b}_{12} - \frac{1}{A}\int \mathbf{b}_{12}\, dA \tag{8.17}$$

The development of this equation is presented in the chapter on incompatible elements, Equation (6.4).

8.6 STATIC CONDENSATION

{ XE "Static Condensation" }The element 16 by 16 stiffness matrix for the plate bending element with shearing deformations is obtained by numerical integration. Or:

$$\overline{\mathbf{K}} = \int \mathbf{B}^T \mathbf{D} \mathbf{B} \, dA = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \tag{8.18}$$

where $\mathbf{K}_{22}$ is the 4 by 4 matrix associated with the incompatible normal rotations. The element equilibrium equations are of the following form:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \tag{8.19}$$

where **F** is the 12 node forces. Because the forces associated with $\Delta\theta$ must be zero, those deformation degrees-of-freedom can be eliminated, by static condensation, before assembly of the global stiffness matrix. Therefore, the 12 by 12 element stiffness matrix is not increased in size if shearing deformations are included. This quadrilateral (or triangular) plate bending element, including shear deformations, is defined in this book as the <u>D</u>iscrete <u>S</u>hear <u>E</u>lement, or DSE.

8.7 TRIANGULAR PLATE BENDING ELEMENT

{ XE "Plate Bending Elements:Triangular Element" **}**The same approximations used to develop the quadrilateral element are applied to the triangular plate bending element with three mid-side nodes. The resulting stiffness matrix is 9 by 9. Approximately 90 percent of the computer program for the quadrilateral element is the same as for the triangular element. Only different shape functions are used and the constraint associated with the fourth side is skipped. In general, the triangle is stiffer than the quadrilateral.

8.8 OTHER PLATE BENDING ELEMENTS

The fundamental equation for the discrete shear along the sides of an element is given by Equation (8.6). Or:

$$\gamma_{ij} = \frac{1}{L}(u_{zj} - u_{zi}) - \frac{1}{2}(\theta_i + \theta_j) - \frac{2}{3}\Delta\theta \tag{8.20}$$

{ XE "Plate Bending Elements:PQ2" **}**If $\Delta\theta$ is set to zero at the mid-point of each side, shearing deformations are still included in the element. However, the internal moments within the element are constrained to a constant value for a thin plate. This is the same as the PQ2 element given in reference [1], which is based on a second order polynomial approximation of the normal displacement. The displacements produced by this element tend to have a small error; however, the internal moments for a coarse mesh tend to have a significant error. Therefore, this author does not recommend the use of this element.

If the shear is set to zero along each side of the element, the following equation is obtained:

$$\Delta\theta = \frac{3}{2L}(w_j - w_i) - \frac{3}{4}(\theta_i + \theta_j) \tag{8.21}$$

Hence, it is possible to directly eliminate the mid-side relative rotations directly without using static condensation. This approximation produces the Discrete Kirchhoff Element, DKE, in which transverse shearing deformations are set to zero. It should be noted that the DSE and the DKE for thin plates converge at approximately the same rate for both displacements and moments. For many problems, the DSE and the DKE tend to be more flexible than the exact solution.

8.9 NUMERICAL EXAMPLES

{ XE "Plate Bending Elements:Convergence" }{ XE "Plate Bending Elements:Examples" }Several examples are presented to demonstrate the accuracy and convergence properties of quadrilateral and triangular plate bending elements with and without transverse shear deformations. A four-point numerical integration formula is used for the quadrilateral element. A three-point integration formula is used for the triangular element.

8.9.1 One Element Beam

To illustrate that the plate element reduces to the same behavior as classical beam theory, the cantilever beam shown in Figure 8.5 is modeled as one element that is 2 inches thick. The narrow element is 6 inches by 0.2 inch in plan.

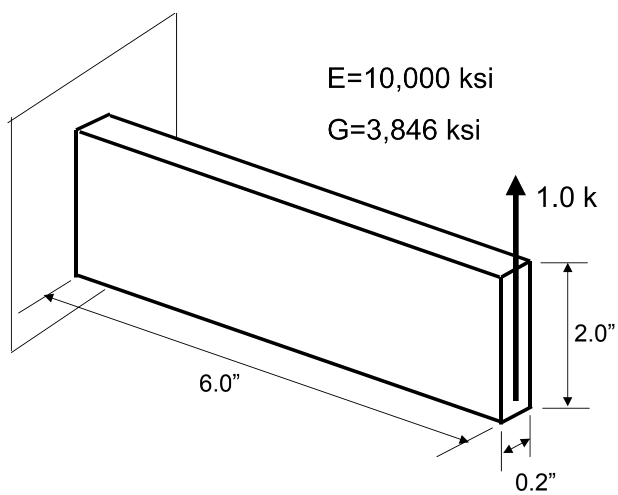

Figure 8.5 Cantilever Beam Modeled using One Plate Element

{ XE "Plate Bending Elements:DKE" }{ XE "Plate Bending Elements:DSE" }The end displacements and base moments are summarized in Table 8.1 for various theories.

Table 8.1 Displacement and Moment for Cantilever Beam

THEORY and ELEMENT	Tip Displacement (inches)	Maximum Moment (kip-in.)
Beam Theory	0.0000540	6.00
Beam Theory with Shear Deformation	0.0000587	6.00
DSE Plate Element	0.0000587	6.00
DKE Plate Element	0.0000540	6.00
PK2 Plate Element – Ref. [1]	0.0000452	3.00

This example clearly indicates that one plate element can model a one-dimensional beam without the loss of accuracy. It is worth noting that many plate elements with shear deformations, which are currently used within computer programs, have the same accuracy as the PQ2 element. Hence, the user must verify the theory and accuracy of all elements within a computer program by checking the results with simple examples.

8.9.2 Point Load On Simply Supported Square Plate

{ XE "Plate Bending Elements:Point Load" }To compare the accuracy of the DSE and DKE as the elements become very thin, a 4 by 4 mesh, as shown in Figure 8.6, models one quadrant of a square plate. Note that the normal rotation along the pinned edge is set to zero. This "hard" boundary condition is required for the DSE. The DKE yields the same results for both hard and soft boundary conditions at the pinned edge.

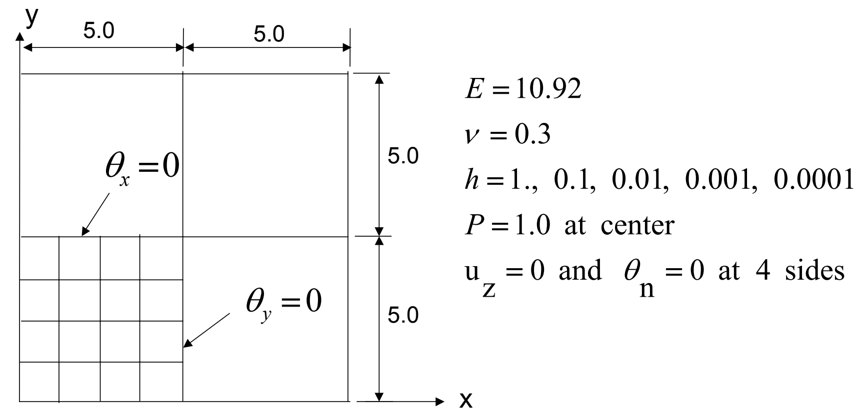

Figure 8.6 Point Load at Center of Simply Supported Square Plate

The maximum displacement and moment at the center of the plate are summarized in Table 8.2. For a thin plate without shear displacements, the displacement is proportional to $1/h^3$. Therefore, to compare results, the displacement is normalized by the factor h^3. The maximum moment is not a function of thickness for a thin plate. For this example, shearing deformations are only significant for a thickness of 1.0. The exact thin-plate displacement for this problem is 1.160, which is very close to the average of the DKE and the DSE results. Hence, one can conclude that DSE converges to an approximate thin plate solution as the plate becomes thin. However, DSE does not converge for a coarse mesh to the same approximate value as the DKE.

Table 8.2 Convergence of Plate Elements – 4 by 4 Mesh – Point Load

Thickness, h	Displacement times h^3		Maximum Moment	
	DKE	DSE	DKE	DSE
1	1.195	1.383	0.3545	0.4273
0.1	1.195	1.219	0.3545	0.4269
0.01	1.195	1.218	0.3545	0.4269
0.001	1.195	1.218	0.3545	0.4269
0.0001	1.195	1.218	0.3545	0.4269

To demonstrate that the two approximations converge for a fine mesh, a 16 by 16 mesh is used for one quadrant of the plate. The results obtained are summarized in Table 8.3.

Table 8.3 Convergence of Plate Element –16 by 16 Mesh – Point Load

Thickness *h*	Displacement times h^3		Maximum Moment	
	DKE	DSE	DKE	DSE
1	1.163	1.393	0.5187	0.5704
0.01	1.163	1.164	0.5187	0.5295
0.0001	1.163	1.164	0.5187	0.5295

One notes that the DKE and DSE displacements converge to the approximately same value for a point load at the center of the plate. However, because of stress singularity, the maximum moments are not equal, which is to be expected.

8.9.3 Uniform Load On Simply Supported Square Plate

To eliminate the problem associated with the point load, the same plate is subjected to a uniform load of 1.0 per unit area. The results are summarized in Table 8.4. For thin plates, the quadrilateral DKE and DSE displacements and moments agree to three significant figures.

Table 8.4 Convergence of Quad Plate Elements –16 by 16 Mesh - Uniform Load

Thickness *h*	Displacement times h^3		Maximum Moment	
	DKE	DSE	DKE	DSE
1	9.807	10.32	1.142	1.144
0.01	9.807	9.815	1.142	1.144
0.0001	9.807	9.815	1.142	1.144

8.9.4 Evaluation of Triangular Plate Bending Elements

The accuracy of the triangular plate bending element can be demonstrated by analyzing the same square plate subjected to a uniform load. The plate is modeled using 512 triangular elements, which produces a 16 by 16 mesh, with each quadrilateral divided into two triangles. The results are summarized in Table 8.5. For thin plates, the quadrilateral DKE and DSE displacements and moments agree to four significant figures. The fact that both moments and displacements converge to the same value for thin plates indicates that the triangular elements may be more accurate than the quadrilateral elements for both thin and thick plates. However, if the triangular mesh is changed by dividing the quadrilateral on the other diagonal the results are not as impressive.

Table 8.5 Convergence of Triangular Plate Elements –- Uniform Load

Thickness *h*	Displacement times h^3		Maximum Moment	
	DKE	DSE	DKE	DSE
1	9.807	10.308	1.145	1.145
0.01	9.807	9.807	1.145	1.145
0.0001	9.807	9.807	1.145	1.145
0.0001*	9.800	9.807	1.142	1.145

* Quadrilateral divided on other diagonal

It should be noted, however, that if the triangular element is used in shell analysis, the membrane behavior of the triangular shell element is very poor and inaccurate results will be obtained for many problems.

8.9.5 Use of Plate Element to Model Torsion in Beams

{ XE "Plate Bending Elements:Torsion" }For one-dimensional beam elements, the plate element can be used to model the shear and bending behavior. However, plate elements should not be used to model the torsional behavior of beams. To illustrate the errors introduced by this approximation, consider the cantilever beam structure shown in Figure 8.7 subjected to a unit end torque.

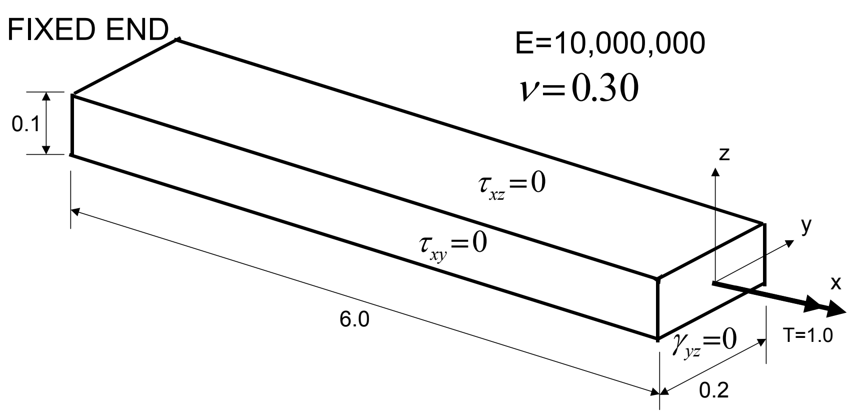

Figure 8.7 Beam Subjected to Torsion Modeled by Plate Elements

The results for the rotation at the end of the beam are shown in Table 8.6.

Table 8.6 Rotation at End of Beam Modeled using Plate Elements

Y-ROTATION	DKE		DSE	
	1 x 6	9 x 9	1 x 6	9 x 9

free	0.0284	0.0233	0.2368	0.1249
fixed	0.0227	0.0218	0.0849	0.0756

The exact solution, based on an elasticity theory that includes warpage of the rectangular cross section, is 0.034 radians. Note that the shear stress and strain boundary conditions shown in Figure 8.6 cannot be satisfied exactly by plate elements regardless of the fineness of the mesh. Also, it is not apparent if the y-rotation boundary condition should be free or set to zero

For this example, the DKE element does give a rotation that is approximately 68 percent of the elasticity solution; however, as the mesh is refined, the results are not improved significantly. The DSE element is very flexible for the coarse mesh. The results for the fine mesh are stiffer. Because neither element is capable of converging to the exact results, the torsion of the beam should not be used as a test problem to verify the accuracy of plate bending elements. Triangular elements produce almost the same results as the quadrilateral elements.

8.10 SUMMARY

{ XE "FLOOR Program" }{ XE "SAFE Program" }A relatively new and robust plate bending element has been summarized in this chapter. The element can be used for both thin and thick plates, with or without shearing deformations. It has been extended to triangular elements and orthotropic materials. The plate bending theory was presented as an extension of beam theory and three-dimensional elasticity theory. The DKE and DSE are currently used in the SAFE, FLOOR and SAP2000 programs.

In the next chapter, a membrane element will be presented with three DOF per node, two translations and one rotation normal to the plane. Based on the bending element presented in this chapter and membrane element presented in the next chapter, a general thin or thick shell element is presented in the following chapter.

8.11 REFERENCES

1. { XE "Ibrahimbegovic, Adnan" }Ibrahimbegovic, Adnan. 1993. “Quadrilateral Elements for Analysis of Thick and Thin Plates,” *Computer Methods in Applied Mechanics and Engineering*. Vol. 110 (1993). 195-209.