6.

INCOMPATIBLE ELEMENTS

When Incompatible Elements Were Introduced in 1971, Mathematics Professor Strang of MIT Stated "In Berkeley, Two Wrongs Make a Right"

6.1 INTRODUCTION

{ XE "Irons, Bruce M." }{ XE "Patch Test" }{ XE "Strang, G." }{ XE "Displacement Compatibility" }In the early years of the development of the Finite Element Method, researchers in the fields of Mathematics, Structural Engineering and Structural Mechanics considered that displacement compatibility between finite elements was absolutely mandatory. Therefore, when the author first introduced incompatible displacements into rectangular isoparametric finite elements at a conference in 1971 [1], the method was received with great skepticism by fellow researchers. The results for both displacements and stresses for rectangular elements were very close to the results from the nine-node isoparametric element. The two ***theoretical crimes*** committed were ***displacement compatibility was violated*** and ***the method was not verified with examples using non-rectangular elements*** [2]. As a consequence of these crimes, Bruce Irons introduced the patch test restriction and the displacement compatible requirement was eliminated [3].

{ XE "Taylor, R. L." }{ XE "Jacobian Matrix" }In 1976 a method was presented by Taylor to correct the incompatible displacement mode; he proposed using a constant Jacobian during the integration of the incompatible modes so that the incompatibility elements passed the patch test [4]. However, the results produced by the non-rectangular isoparametric element were not impressive.

{ XE "B Bar Method" }{ XE "Rafai, M.S." }{ XE "Incompatible Displacements" }{ XE "Simo, J. C." }In 1986 Simo and Rafai introduced the B bar method to correct the strains produced by incompatible displacements, achieving excellent results for non-rectangular elements [5]. Since that time the use of incompatible lower-order elements has reduced the need for reduced integration and the use of very high-order isoparametric elements. Many of these new elements, based on corrected incompatible displacement modes, are summarized in this book.

6.2 ELEMENTS WITH SHEAR LOCKING

{ XE "Shear Locking" }The simple four-node isoparametric element does not produce accurate results for many applications. To illustrate this deficiency, consider the rectangular element, shown in Figure 6.1, subjected to pure bending

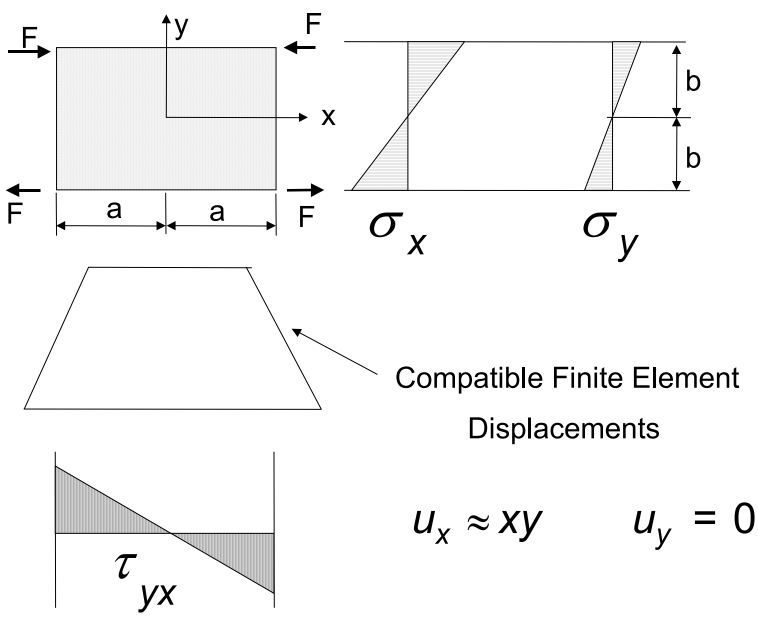

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Figure 6.1 Basic Equilibrium Errors in Four-Node Plane Element

{ XE "Poisson's Ratio" }It is apparent that the compatible four-point rectangular element produces significant errors in both displacements and stresses when subjected to simple stress gradients. Shear-locking is the term used to describe the development of shear stresses when the element is subjected to pure bending. In addition to the shear stress problem, an error in the vertical stress is developed because of the Poisson's ratio effect. The exact displacements, which allow the element to satisfy internal equilibrium, are of the form:

$$u_x = c_1 xy \quad \text{and} \quad u_y = c_2 (1 - \left(\frac{x}{a}\right)^2) + c_3 (1 - \left(\frac{y}{b}\right)^2) \tag{6.1}$$

These displacements allow the shear strain to be zero at all points within the element. Also, the neutral axis must move vertically, thereby reducing the vertical stresses to zero.

6.3 ADDITION OF INCOMPATIBLE MODES

The motivation for the addition of incompatible displacement modes, of magnitude α_j, is to cancel the stresses associated with the error terms defined in Equation (6.1). Or, in terms of the r-s natural reference system, the new displacement shape functions for the four-node isoparametric element are:

$$\begin{aligned} u_x &= \sum_{i=1}^{4} N_i u_{xi} + \alpha_1 (1 - r^2) + \alpha_2 (1 - s^2) \\ u_y &= \sum_{i=1}^{4} N_i u_{xy} + \alpha_3 (1 - r^2) + \alpha_4 (1 - s^2) \end{aligned} \tag{6.2}$$

Hence, the strain-displacement equation for an incompatible element can be written as:

$$\mathbf{d} = \begin{bmatrix} \mathbf{B_C} & \mathbf{B_I} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \alpha \end{bmatrix} \tag{6.3}$$

If we let $\mathbf{d}^\mathbf{T} = \lfloor \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \rfloor$ and $\mathbf{f}^\mathbf{T} = \lfloor \sigma_x \quad \sigma_y \quad \tau_{xy} \rfloor$, the strain energy within the incompatible element is given as:

$$W = \frac{1}{2}\int \mathbf{f}^{\mathbf{T}}\mathbf{d}\, dV = \frac{1}{2}\int \mathbf{f}^{\mathbf{T}}\mathbf{B}_{\mathbf{C}}\,\mathbf{u}\, dV + \frac{1}{2}\int \mathbf{f}^{\mathbf{T}}\mathbf{B}_{\mathbf{I}}\,\alpha\, dV \tag{6.4}$$

To pass the patch test, the strain energy associated with the incompatible modes must be zero for a state of constant element stress. Hence, for a state of constant stress, the following equation must be satisfied:

$$\frac{1}{2}\mathbf{f}^{\mathbf{T}}\int \mathbf{B}_{\mathbf{I}}\,\alpha\, dV = 0 \quad \text{or} \quad \int \mathbf{B}_{\mathbf{I}}\; dV = 0 \tag{6.5}$$

{ XE "Correction Matrix" }This can be satisfied if we add a constant ***correction matrix*** $\mathbf{B}_{\mathbf{IC}}$ to the $\mathbf{B}_{\mathbf{I}}$ matrix and to form a new strain-displacement, $\overline{\mathbf{B}}_{\mathbf{I}} = \mathbf{B}_{\mathbf{I}} + \mathbf{B}_{\mathbf{IC}}$, so that the following equation is satisfied:

$$\int (\mathbf{B}_{\mathbf{I}} + \mathbf{B}_{\mathbf{IC}})\, dV = 0 \;\text{ or,}\quad \int \mathbf{B}_{\mathbf{I}}\; dV + V\,\mathbf{B}_{\mathbf{IC}} = 0 \tag{6.6}$$

The volume of the element is V. Hence, the correction matrix can be calculated from:

$$\mathbf{B}_{\mathbf{IC}} = -\frac{1}{V}\int \mathbf{B}_{\mathbf{I}}\; dV \tag{6.7}$$

This is a very general approach and can be used to add any number of incompatible displacement modes, or strain patterns, to all types of isoparametric elements. The same numerical integration formula should be used to evaluate Equation (6.7) as is used in calculating the element stiffness matrix.

6.4 FORMATION OF ELEMENT STIFFNESS MATRIX

In the minimization of the potential energy the forces associated with the incompatible displacement modes α are zero. Therefore, the element equilibrium equations are given by:

$$\begin{bmatrix} \mathbf{f}_{\mathbf{c}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{\mathbf{CC}} & \mathbf{k}_{\mathbf{CI}} \\ \mathbf{k}_{\mathbf{IC}} & \mathbf{k}_{\mathbf{II}} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \alpha \end{bmatrix} \tag{6.8}$$

The individual sub-matrices within the element stiffness matrix are given by:

$$\mathbf{k_{CC}} = \int \mathbf{B_C^T E B_C}\ dV \tag{6.9a}$$

$$\mathbf{k_{CI}} = \int \mathbf{B_C^T E \overline{B}_I}\ dV \tag{6.9b}$$

$$\mathbf{k_{IC}} = \int \mathbf{\overline{B}_I^T E B_C}\ dV \tag{6.9c}$$

$$\mathbf{k_{II}} = \int \mathbf{\overline{B}_I^T E \overline{B}_I}\ dV \tag{6.9d}$$

{ XE "Static Condensation" }Using ***static condensation*** [6] the incompatible displacement modes are eliminated before assembly of the element stiffness matrices. Or:

$$\mathbf{f_C} = \mathbf{k_C}\,\mathbf{u} \tag{6.10}$$

Therefore, the element stiffness matrix is given by:

$$\mathbf{k_C} = \mathbf{k_{CC}} - \mathbf{k_{CI}}\,\mathbf{k_{II}^{-1}}\,\mathbf{k_{IC}} \tag{6.11}$$

Symbolically, Equation (6.11) is correct; however, it should be pointed out that matrix inversion and matrix multiplication are not used in the static condensation algorithm as presented in Section 4.5 for the modification of frame element stiffness because of moment end releases.

6.5 INCOMPATIBLE TWO-DIMENSIONAL ELEMENTS

The addition of the incompatible shape functions, $(1-s^2)$ and $(1-r^2)$, to u_x and u_y displacement approximations is very effective for plane rectangular elements. Therefore, for quadrilaterals of arbitrary shape, the following displacement approximation has been found to be effective:

$$\begin{aligned} u_x &= \sum_{i=1}^{4} N_i u_{xi} + \sum_{i=5}^{6} N_i \alpha_{xi} \\ u_y &= \sum_{i=1}^{4} N_i u_{xy} + \sum_{i=5}^{6} N_i \alpha_{yi} \end{aligned} \tag{6.12}$$

The incompatible shape functions are:

$$N_5 = 1 - r^2$$
$$N_6 = 1 - s^2 \tag{6.13}$$

The four incompatible modes increase computational time required to form the element stiffness matrix; however, the improvement in accuracy is worth the additional calculations.

6.6 EXAMPLE USING INCOMPATIBLE DISPLACEMENTS

{ XE "Incompatible Displacements" }To illustrate the accuracy of both compatible and incompatible elements in two dimensions, the cantilever beam shown in Figure 6.2 is analyzed assuming a moment and concentrated forces acting at the end of the cantilever.

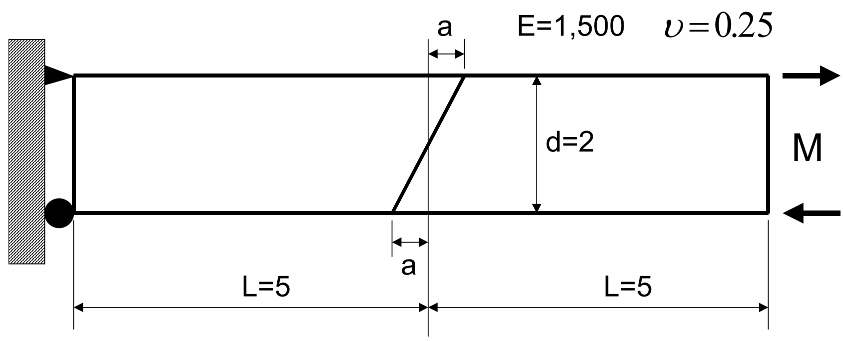

Figure 6.2 Beam Modeled with Distorted Mesh

An element shape sensitivity study can be accomplished using different distortion factors. Table 6.1 presents a summary of the results.

Table 6.1 Results of Analysis of Cantilever Beam

Mesh Distortion Factor “a”	Number of Incompatible Modes	TIP MOMENT LOADING		TIP SHEAR LOADING	
		Normalized Tip Displacement	Normalized Maximum Stress At Support	Normalized Tip Displacement	Normalized Maximum Stress At Support
EXACT	-	1.000	1.000	1.000	1.000
0	0	0.280	0.299	0.280	0.149
0	4	1.000	1.000	0.932	0.750
1	4	0.658	0.638	0.706	0.600
2	4	0.608	0.657	0.688	0.614

It is apparent that the classical four-node, rectangular, compatible isoparametric element, without incompatible modes, produces very poor results. The use of this ***classical element*** can produce significant errors that may have serious practical engineering consequences. One notes that the stresses may be less than 20 percent of the correct value.

The addition of four parabolic shape functions produces the exact values of displacements and stresses for rectangular elements resulting from constant moment loading. However, because of tip shear loading, the maximum stress has a 25 percent error. In addition, as the element is distorted, the accuracy of both displacements and stresses is reduced by 30 to 40 percent.

It should be noted that all elements pass the patch test and will converge to the exact solution, as the mesh is refined. It appears that the plane quadrilateral elements, with eight incompatible displacement modes, will converge faster than the lower-order elements.

6.7 THREE-DIMENSIONAL INCOMPATIBLE ELEMENTS

The classical eight-node, hexahedral displacement compatible element has the same shear-locking problem as the classical, four-node plane element. The addition of nine incompatible shape functions has proven effective for three dimensional, eight-node, hexahedral elements. Or:

$$
\begin{aligned}
u_x &= \sum_{i=1}^{8} N_i u_{xi} + \sum_{9}^{11} N_i a_{xi} \\
u_y &= \sum_{i=1}^{8} N_i u_{yi} + \sum_{9}^{11} N_i a_{yi} \\
u_z &= \sum_{i=1}^{8} N_i u_{xi} + \sum_{9}^{11} N_i a_{yi}
\end{aligned}
\tag{6.14}
$$

The three additional incompatible shape functions are:

$$
\begin{aligned}
N_9 &= 1 - r^2 \\
N_{10} &= 1 - s^2 \\
N_{11} &= 1 - t^2
\end{aligned}
\tag{6.15}
$$

The 2 by 2 by 2 integration formula previously presented for three-dimensional isoparametric elements has been found to be effective for the eight-node hexahedral element with nine additional incompatible modes.

6.8 SUMMARY

Because of the serious problem associated with shear-locking, the classical compatible four-node quadrilateral and eight-node hexahedral elements should not be used to simulate the behavior of real structures. It has been demonstrated that the addition of incompatible displacement modes, corrected to pass the patch test, significantly enhances the performance of quadrilateral and hexahedral isoparametric elements.

The nine-node quadrilateral and the 27-node hexahedral elements are accurate and can be improved by adding corrected incompatible modes. For example, cubic modes can be added to the nine-node plane element in which the exact results can be calculated, for tip shear loading, using only one element to model a cantilever beam [7].

6.9 REFERENCES

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