

4.

ONE-DIMENSIONAL ELEMENTS

Before 1960, the Field of Structural Analysis Was Restricted to One-Dimensional Elements

4.1 INTRODUCTION

{ XE "Beams" } { XE "Frame Element" } { XE "Non-Prismatic Element" } Most structural engineers have the impression that two- and three-dimensional finite elements are very sophisticated and accurate compared to the one-dimensional frame element. After more than forty years of research in the development of practical structural analysis programs, it is my opinion that *the non-prismatic frame element, used in an arbitrary location in three-dimensional space, is definitely the most complex and useful element compared to all other types of finite elements.*

{ XE "Arbitrary Frame Element" } The fundamental theory for frame elements has existed for over a century. However, only during the past forty years have we had the ability to solve large three-dimensional systems of frame elements. In addition, we now routinely include torsion and shear deformations in all elements. In addition, the finite size of connections is now considered in most analyses. Since the introduction of computer analysis, the use of non-prismatic sections and arbitrary member loading in three-dimensions has made the programming of the element very tedious. In addition, the post processing of the frame forces to satisfy the many different building codes is complex and not clearly defined.

4.2 ANALYSIS OF AN AXIAL ELEMENT

{ XE "Axial Element" } To illustrate the application of the basic equations presented in the previous chapter, the 2 x 2 element stiffness matrix will be developed for the truss element shown in Figure 4.1.

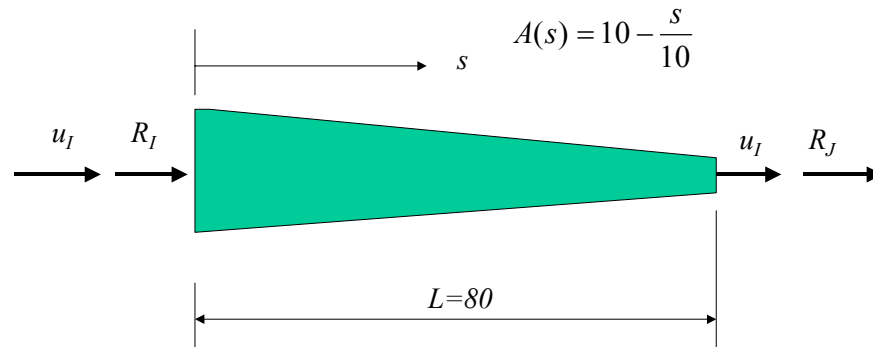


Figure 4.1 Tapered Bar Example

The axial displacements at position s can be expressed in terms of the axial displacements at points I and J at the ends of the element. Or:

$$u(s) = u_I + \frac{s}{L}(u_J - u_I) \quad (4.1)$$

The axial strain is by definition $\epsilon_s = \frac{\partial u}{\partial s}$. Hence, the strain-displacement relationship will be:

$$\epsilon = \frac{1}{L}(u_J - u_I) = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} u_I \\ u_J \end{bmatrix} = \mathbf{B} \mathbf{u} \quad (4.2)$$

The stress-strain relationship is $\sigma = E\epsilon$. Therefore, the element stiffness matrix is:

$$\mathbf{k}^{(i)} = \int \mathbf{B}^{(i)T} \mathbf{E}^{(i)} \mathbf{B}^{(i)} dV = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (4.3)$$

Because the strain is constant, integration over the element produces the volume $A_a L$ where A_a is the average cross-sectional area of the element. If the cross-sectional area is constant, the stiffness matrix is exact and the force and displacement methods will produce identical results. However, if the area is not constant, significant errors may be introduced by the formal application of the displacement method.

To illustrate the errors involved in the application of the displacement method, let us assume the following properties:

$$E=1,000 \text{ ksi} \quad A_a = 6.0 \text{ in}^2 \quad L = 80 \text{ in.} \quad u_I = 0 \quad R_J = 10 \text{ kips}$$

{ XE "Assumed Displacement" } Hence, the displacement at point J is given by:

$$u_J = \frac{L}{A_a E} R_J = 0.1333 \text{ in.} \quad (4.4)$$

{ XE "Assumed Stress" } From equation (4.2), the corresponding constant strain is 0.0016666. Therefore, the constant axial stress is given by:

$$\sigma = E\varepsilon = 1.667 \text{ ksi} \quad (4.5)$$

However, if a force approach is used for the solution of this problem, significant and more accurate results are obtained. From simple statics, the axial stress distribution is:

$$\sigma = \frac{R_J}{A(s)} = \frac{10}{100-s} R_J = \mathbf{P R} \quad (4.6)$$

From the force method, the displacement at the end of the member is given by:

$$u_J = \left[\int \mathbf{P}^T \mathbf{C P} dV \right] \mathbf{R} = \frac{1}{E} \int_0^{80} \frac{10}{100-s} ds \mathbf{R} = 0.1607 \text{ in.} \quad (4.7)$$

Note that the end displacement obtained by the displacement method is approximately 17 percent less than the exact displacement produced by the force method.

Of greater significance, however, is the comparison of the axial stress distribution summarized in Figure 4.2, using both the force and displacement methods of analysis.

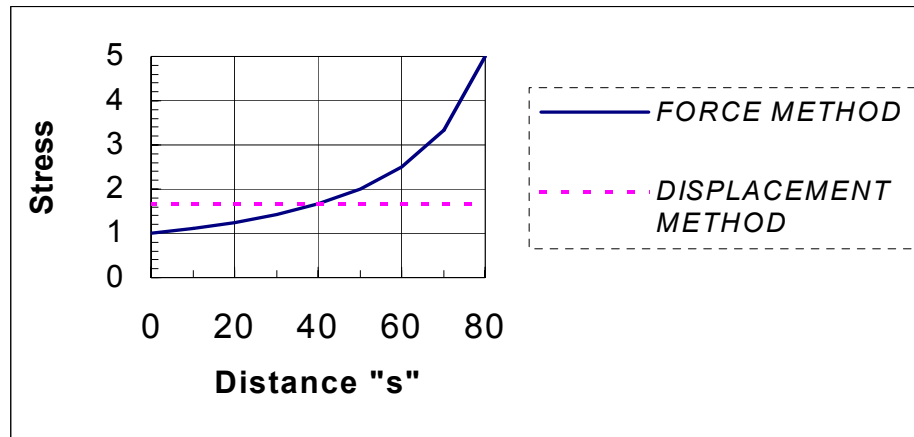


Figure 4.2 Comparison of Stresses for Force and Displacement Method

At the end of the tapered rod, the displacement method produces only 33 percent of the maximum stress of 5.0 ksi. Of course, if a fine mesh is used, the results will be closer. Also, if higher order elements are used, with interior points, the displacement method results can be improved significantly. Nevertheless, this example clearly illustrates that *the force approach should be used to predict the behavior of one-dimensional elements.*

4.3 TWO-DIMENSIONAL FRAME ELEMENT

{ XE "Rigid Body Displacements" } A non-prismatic frame element with axial, bending and shearing deformations will be developed to illustrate the power of the force method. The displacement method has the ability to calculate a stiffness matrix of any element directly in terms of all displacement degrees-of-freedom associated with the elements; and the element automatically includes the rigid body displacement modes of the element. The force method only allows for the development of the element flexibility matrix in terms of displacements relative to a stable support system.

The general frame element is composed of any number of non-prismatic frame segments. Each segment can have independent axial, shear or bending properties. Therefore, at the ends of the element, rigid bending segments are possible, with or without shearing and axial deformations. Hence, it is possible to approximate the behavior of the finite connection area. A typical frame member is shown in Figure 4.3.

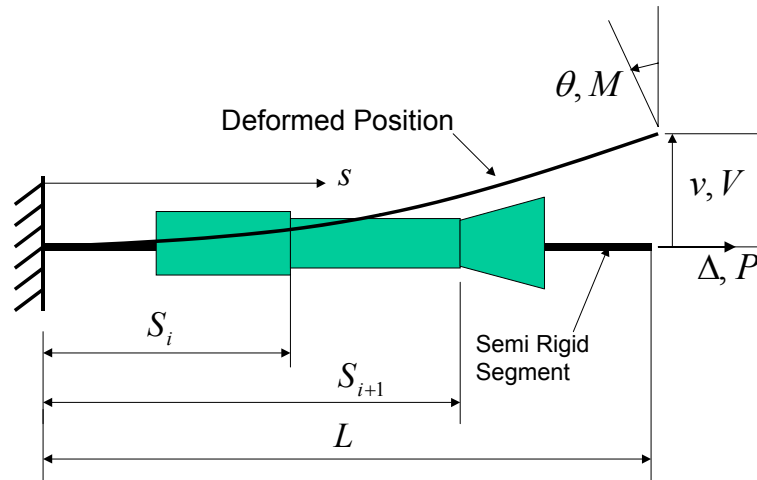


Figure 4.3 Arbitrary, Two-Dimensional Frame Element

The relative displacements are the axial displacement Δ , vertical displacement v , and the end rotation θ . The corresponding loads are the axial load P , vertical load V , and the end moment M . At a typical cross-section at location s , the force-deformation relationship is:

$$\mathbf{d}(s) = \mathbf{C}(s) \mathbf{f}(s), \text{ or } \begin{bmatrix} \varepsilon(s) \\ \gamma(s) \\ \psi(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{E(s)A(s)} & 0 & 0 \\ 0 & \frac{1}{G(s)A_s(s)} & 0 \\ 0 & 0 & \frac{1}{E(s)I(s)} \end{bmatrix} \begin{bmatrix} P(s) \\ V(s) \\ M(s) \end{bmatrix} \quad (4.8)$$

All cross-sectional properties, including the effective shear area A_s , can vary within each segment of the frame element.

{ XE "Section Forces" } { XE "Section Properties" } The section forces within a typical segment at location s can be expressed directly from statics in terms of the arbitrary end forces \mathbf{R} . Or:

$$\mathbf{f}(s) = \mathbf{P}(s)\mathbf{R}, \text{ or } \begin{bmatrix} P(s) \\ V(s) \\ M(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L-s & 1 \end{bmatrix} \begin{bmatrix} P \\ V \\ M \end{bmatrix} \quad (4.9)$$

The 3 x 3 flexibility matrix as defined by the force method is calculated from:

$$\mathbf{F} = \int_0^L \mathbf{P}(s)^T \mathbf{C}(s) \mathbf{P}(s) ds = \sum_i^{I_{MAX}} \int_{S_i}^{S_{i+1}} \mathbf{P}(s)^T \mathbf{C}(s) \mathbf{P}(s) ds \quad (4.10)$$

It is of interest to note that because of the discontinuity of the properties of the segments, each segment produces a separate 3 by 3 flexibility matrix. Therefore, Equation (4.10) can be written in the following form:

$$\mathbf{F} = \sum_i^{I_{MAX}} \mathbf{F}^{(i)}, \text{ where } \mathbf{F}^{(i)} = \int_{S_i}^{S_{i+1}} \mathbf{P}(s)^T \mathbf{C}(s) \mathbf{P}(s) ds \quad (4.11)$$

{ XE "Direct Flexibility Method" } Equation (4.11) can be termed the *direct flexibility method*, because the segment flexibility terms are directly added. It should be pointed out that if any cross-sectional stiffness properties are infinite, as defined in Equation (4.9), the contribution to the flexibility at the end of the element is zero.

The \mathbf{C} and \mathbf{P} matrices contain a significant number of zero terms. Therefore, the element flexibility matrix for a straight member contains only four independent terms, which are illustrated by:

$$\mathbf{F} = \begin{bmatrix} F_P & 0 & 0 \\ 0 & F_{VV} & F_{VM} \\ 0 & F_{VM} & F_{MM} \end{bmatrix}, \quad (4.12)$$

It can easily be shown that the individual flexibility terms are given by the following simple equations:

$$F_P = \sum_i^{I_{MAX}} \int_{S_i}^{S_{i+1}} \frac{1}{E(s)A(s)} ds \quad (4.13a)$$

$$F_{VV} = \sum_i^{I_{MAX}} \int_{S_i}^{S_{i+1}} \left[\frac{(L-s)^2}{E(s)I(s)} + \frac{1}{G(s)A_s(s)} \right] ds \quad (4.13b)$$

$$F_{VM} = \sum_i^{I_{MAX}} \int_{S_i}^{S_{i+1}} \frac{(L-s)}{E(s)I(s)} ds \quad (4.13c)$$

$$F_{MM} = \sum_i^{I_{MAX}} \int_{S_i}^{S_{i+1}} \frac{1}{E(s)I(s)} ds \quad (4.13d)$$

{ XE "Frame Element:Properties" } For frame segments with constant or linear variation of element properties, those equations can be evaluated in closed form. For the case of more complex segment properties, numerical integration may be required. For a prismatic element without rigid end offsets, those flexibility constants are well-known and reduce to:

$$F_P = \frac{L}{EA} \quad (4.14a)$$

$$F_{VV} = \frac{L^3}{3EI} + \frac{L}{GA_s} \quad (4.14b)$$

$$F_{VM} = \frac{L^2}{2EI} \quad (4.14c)$$

$$F_{MM} = \frac{L}{EI} \quad (4.14d)$$

For rectangular cross-sections, the shear area is $A_s = \frac{5}{6} A$.

One can easily consider loading within the segment by calculating the additional relative displacements at the end of the element using simple virtual work methods. For this more general case, the total relative displacement will be of the following form:

$$\begin{bmatrix} \Delta \\ v \\ \theta \end{bmatrix} = \begin{bmatrix} F_P & 0 & 0 \\ 0 & F_{VV} & F_{VM} \\ 0 & F_{VM} & F_{MM} \end{bmatrix} \begin{bmatrix} P \\ V \\ M \end{bmatrix} + \begin{bmatrix} \Delta_L \\ v_L \\ \theta_L \end{bmatrix} \quad \text{or, } \mathbf{v} = \mathbf{FR} + \mathbf{v}_L \quad (4.15)$$

The displacements caused by span loading are designated by \mathbf{v}_L . Equation (4.15) can be rewritten in terms of the element stiffness as:

$$\mathbf{r} = \mathbf{Kv} - \mathbf{Kv}_L = \mathbf{Kv} - \mathbf{r}_L \quad (4.16)$$

The element stiffness is the inverse of the element flexibility, $\mathbf{K} = \mathbf{F}^{-1}$, and the fixed-end forces caused by span loading are $\mathbf{r}_L = \mathbf{Kv}_L$. Within a computer program, those equations are evaluated numerically for each element; therefore, it is not necessary to develop the element stiffness in closed form.

4.4 THREE-DIMENSIONAL FRAME ELEMENT

{ XE "Shearing Deformations" } { XE "Torsional Flexibility" } The development of the three-dimensional frame element stiffness is a simple extension of the equations presented for the two-dimensional element. Bending and shearing deformations can be included in the normal direction using the same equations. In addition, it is apparent that the uncoupled torsional flexibility is given by:

$$F_T = \sum_i^{I_{MAX}} \int_{S_i}^{S_{i+1}} \frac{1}{G(s)J(s)} ds \quad (4.17)$$

{ XE "Torsional Stiffness" } The torsional stiffness term, $G(s)J(s)$, can be difficult to calculate for many cross-sections. The use of a finite element mesh may be necessary for complex sections.

An arbitrary, three-dimensional frame element is shown in Figure 4.4. Note that only the six forces at the J end are shown. The six relative displacements at node J have the same positive sign convention as the forces at node J.

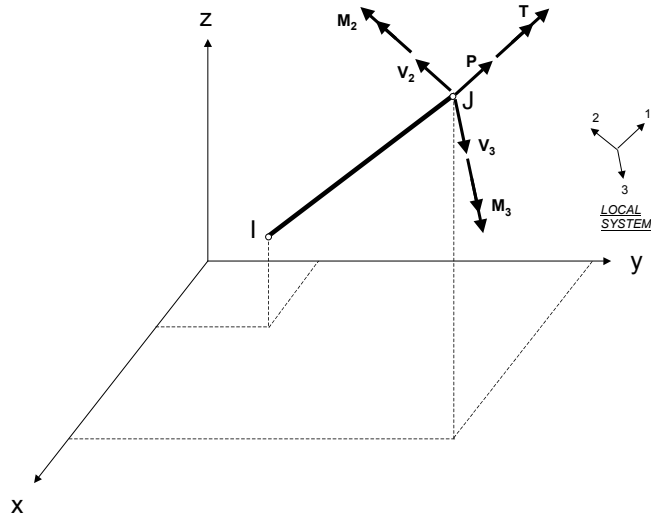


Figure 4.4 Member Forces in Local Reference Systems

{ XE "Frame Element:Local Reference System" }The 6 by 6 stiffness matrix is formed in the local 1-2-3 coordinate system, as shown in Figure 4.4. The order of the forces and relative deformations are given by:

$$\begin{bmatrix} P \\ V_2 \\ V_3 \\ T \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{k}_{22} & 0 & 0 & 0 & \mathbf{k}_{26} \\ 0 & 0 & k_{33} & 0 & k_{35} & 0 \\ 0 & 0 & 0 & k_{44} & 0 & 0 \\ 0 & 0 & k_{53} & 0 & k_{55} & 0 \\ 0 & \mathbf{k}_{62} & 0 & 0 & 0 & \mathbf{k}_{66} \end{bmatrix} \begin{bmatrix} \Delta \\ v_2 \\ v_3 \\ \phi_T \\ \theta_2 \\ \theta_3 \end{bmatrix} \quad \text{or, } \mathbf{f}_J = \mathbf{k}_J \mathbf{d}_J \quad (4.18)$$

The bold terms indicate the shear and bending contributions in the 1-2 plane. For a curved member in three dimensions, the 6 by 6 \mathbf{k} matrix may be full without the existence of any zero terms. Note that the 6 by 6 stiffness matrix formed in the local system does not have the six rigid body modes.

The forces acting at node **I** are not independent and can be expressed in terms of the forces acting at node **J** by the application of the basic equations of statics. Therefore:

$$\begin{bmatrix} P \\ V_2 \\ V_3 \\ T \\ M_2 \\ M_3 \end{bmatrix}_I = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \frac{1}{L} \\ 0 & 0 & -1 & 0 & \frac{1}{L} & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & L & 0 & -1 & 0 \\ 0 & L & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} P \\ V_2 \\ V_3 \\ T \\ M_2 \\ M_3 \end{bmatrix}_J \quad \text{or, } \mathbf{f}_I = \mathbf{b}_{IJ}^T \mathbf{f}_J \quad (4.19)$$

The twelve forces at both ends of the beam can now be expressed in terms of the six forces at the **J** end of the beam by the following submatrix equations:

$$\begin{bmatrix} \mathbf{f}_I \\ \mathbf{f}_J \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{IJ}^T \\ \mathbf{I} \end{bmatrix} \mathbf{f}_J \quad \text{or, } \mathbf{f}_{IJ} = \mathbf{b}^T \mathbf{f}_J \quad (4.20)$$

{ XE "Frame Element:Displacement Transformation" } Also, from the relationship between the equations of statics and compatibility, the following displacement transformation equation exists:

$$\mathbf{d}_I = \mathbf{b} \mathbf{d}_{IJ} \quad (4.21)$$

Therefore, the 12 by 12 frame element stiffness, \mathbf{k}_{IJ} , with respect to the local 1-2-3 reference system, is:

$$\mathbf{k}_{IJ} = \mathbf{b}^T \mathbf{k}_j \mathbf{b} \quad (4.22)$$

Hence, the force-displacement equations in the local 1-2-3 system can be written as:

$$\mathbf{f}_{IJ} = \mathbf{k}_{IJ} \mathbf{u}_{IJ} \quad (4.23)$$

To use the direct stiffness formulation, it is necessary to transform the local element stiffness into the global x-y-z reference system. The global 12 by 12

stiffness matrix must be formed with respect to the node forces shown in Figure 4.5. All twelve node forces \mathbf{R} and twelve node displacements \mathbf{u} have the same sign convention.

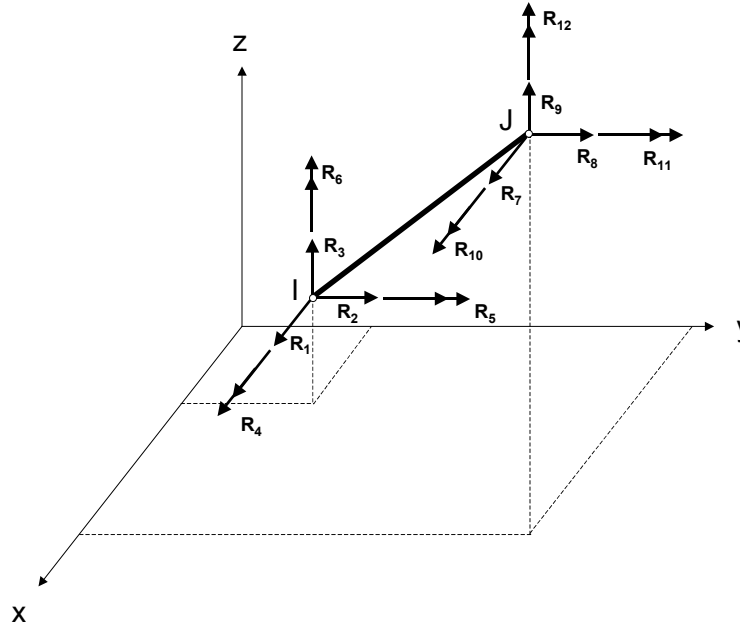


Figure 4.5 Frame Member Forces in Absolute Reference System

{ XE "Frame Element:Absolute Reference System" } The local displacements and forces can be expressed using the elementary direction cosine matrix given in Appendix A. Or:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{V} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \mathbf{V}^T \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \quad (4.24)$$

Therefore, the final twelve transformation equations are in the following simple 4 by 4 submatrix form:

$$\mathbf{u}_{IJ} = \begin{bmatrix} \mathbf{V} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{V} \end{bmatrix} \mathbf{u} \quad \text{or,} \quad \mathbf{u}_{IJ} = \mathbf{T} \mathbf{u} \quad (4.25)$$

The twelve global equilibrium equations in x-y-z reference system are now given by:

$$\mathbf{R} = \mathbf{K}\mathbf{u} + \mathbf{R}_L \quad (4.26)$$

The frame element stiffness matrix is:

$$\mathbf{K} = \mathbf{T}^T \mathbf{k}_{IJ} \mathbf{T} \quad (4.27)$$

It can be shown that the six fixed-end forces \mathbf{r}_j caused by member loads, which are defined in the local 1-2-3 system, can be transformed to twelve global loads by:

$$\mathbf{R}_L = \mathbf{T}^T \mathbf{b}^T \mathbf{r}_j \quad (4.28)$$

It should be pointed out that within most efficient computer programs, formal matrix multiplication is not used to form the matrices. Programming methods are used to skip most multiplication by zero terms.

4.5 MEMBER END-RELEASES

{ XE "Frame Element:Member End Releases" } { XE "Frame Element:Member Loading" } Including member loading in Equation (4.23), the twelve equilibrium equations in the local IJ reference system can be written as

$$\mathbf{f}_{IJ} = \mathbf{k}_{IJ} \mathbf{u}_{IJ} + \mathbf{r}_{IJ} \quad \text{or, without subscripts } \mathbf{f} = \mathbf{k}\mathbf{u} + \mathbf{r} \quad (4.29)$$

If one end of the member has a hinge, or other type of release that causes the corresponding force to be equal to zero, Equation (4.29) requires modification. A typical equation is of the following form:

$$f_n = \sum_{j=1}^{12} k_{nj} u_j + r_n \quad (4.30)$$

If we know a specific value of f_n is zero because of a release, the corresponding displacement u_n can be written as:

$$u_n = \sum_{j=1}^{n-1} \frac{k_{nj}}{k_{nn}} u_j + \sum_{j=n+1}^{12} \frac{k_{nj}}{k_{nn}} u_j + r_n \quad (4.31)$$

Therefore, by substitution of equation (4.31) into the other eleven equilibrium equations, the unknown u_n can be eliminated and the corresponding row and column set to zero. Or:

$$\bar{\mathbf{f}}_{\mathbf{IJ}} = \bar{\mathbf{k}}_{\mathbf{IJ}} \mathbf{u}_{\mathbf{IJ}} + \bar{\mathbf{r}}_{\mathbf{IJ}} \quad (4.32)$$

The terms $f_n = r_n = 0$ and the new stiffness and load terms are equal to:

$$\begin{aligned} \bar{k}_{ij} &= k_{ij} - k_{in} \frac{k_{nj}}{k_{nn}} \\ \bar{r}_i &= r_i - r_n \frac{k_{ni}}{k_{nn}} \end{aligned} \quad (4.33)$$

{ XE "Static Condensation" } { XE "Partial Gauss Elimination" } This procedure can be repeatedly applied to the element equilibrium equations for all releases. After the other displacements associated with the element have been found from a solution of the global equilibrium equations, the displacements associated with the releases can be calculated from Equation (4.31) in reverse order from the order in which the displacements were eliminated. The repeated application of these simple numerical equations is defined in Appendix C as *static condensation* or *partial Gauss elimination*.

4.6 SUMMARY

The force method should be used to develop the stiffness matrices for one-dimensional elements where the internal section stress-resultants can be expressed, by satisfying equilibrium, in terms of the forces acting on the ends of the element. First, the flexibility matrix, with respect to a stable support system, is developed in the element local reference system. Second, this flexibility matrix is inverted to form the element stiffness matrix. Third, the local stiffness matrix is expanded to include the rigid-body displacements and is modified because of

end releases. Finally, the stiffness and load matrices are transformed into the global reference system.