A NATURAL SET OF EIGENVECTORS FOR THE DYNAMIC ANALYSIS OF STRUCTURAL SYSTEMS

Edward L. Wilson and G. Robert Morris

ABSTRACT

A general numerical algorithm is presented that produces a complete set of orthogonal vectors that can be used to reduce the number of unknown displacements required to produce an accurate analysis of structures subjected to both static and dynamic loads. For any specified type of loading error indicators are calculated and the generation of the vectors can be terminated automatically; hence, the engineer/program-user need only to specify the error tolerance and not the number of vectors requested. Therefore, an optimum number of vectors are calculated and there is no need to add approximate static load corrections to the solution or to be concerned about missing mass in a dynamic response analysis. The new algorithm generates a different set of vectors for different types of spatial loading acting on the structure. Thus, the vectors generated are a set of ***N****atural* ***L****oad* ***D****ependent****, NLD****, vectors for the problem to be solved.*

A special application of the NLD vector algorithm, using random load vectors and iteration, produces exacts dynamic eigenvectors, rigid-body modes and static elastic modes for any structural system. However, the use of these exact eigenvectors in a dynamic response analysis may require a very large number of vectors to produce the same accuracy as obtained by the use of a small number of NLD vectors. The NLD vectors are always a linear combination of the exact eigenvectors; however, eigenvectors are not calculated unless they are required to obtain an accurate solution. The algorithm has been applied to the dynamic analysis of many large structural systems with over 50,000 degrees-of-freedom. The method has proven to be very robust, numerically efficient and accurate.

NATURAL EIGENVECTORS OF A STRUCTURAL SYSTEM

In the analysis of structures subjected to three base accelerations there is a requirement that one must include enough modes to account for 90 percent of the mass in the three global directions. However, for other types of loading, such as base displacement loads and point loads, there are no guidelines as to how many modes are to be used in the analysis. In many cases it has been necessary to add static correction vectors to the truncated modal solution in order to obtain accurate results. One of the reasons for these problems is that number of eigenvectors required to obtain an accurate solution is a function of the type of loading that is applied to the structure. However, the major reason for the existence of these numerical problems is that ***all of the natural eigenvectors of the structural system are not included in the analysis***.

In order to illustrate the physical significance of the complete set of natural eigenvectors for a structure consider the unsupported beam shown in Figure 1a. The two-dimensional structure has six displacement DOF, three rotations (each with no rotational mass) and three vertical displacements (each with a vertical lumped mass).

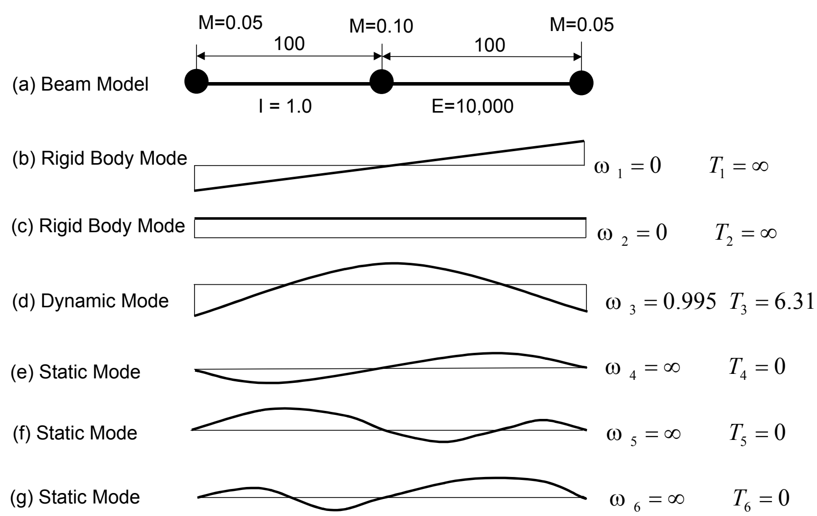

Figure 1. Rigid-Body , Dynamic and Static Modes for Simple Beam

The six stiffness and mass orthogonal natural eigenvectors with frequency ω_n , in radians per second, and period T_n , in seconds, are shown in Figure 1b to 1g. The maximum number of natural eigenvectors that are possible is always equal to the number of displacement DOF. The static vectors (modes) have infinite frequencies; therefore, it is not possible to use the classical definition that the eigenvalues are equal to ω_n^2 if the eigenvalues are to be numerically evaluated. A new definition of the natural eigenvalue and the new algorithm used to numerically evaluate the complete set of natural eigenvectors will be presented in detail later in the paper.

Note that the rigid-body modes only have kinetic energy and the static modes only have strain energy. Whereas, the free vibration dynamic modes contain both kinetic and strain energy – the sum of which at any time is always a constant. Also, the eigenvectors with identical frequencies are not unique vectors. Any linear combination of eigenvectors, with the same frequency, will satisfy the orthogonality requirements.

STRUCTURAL EQUILIBRIUM EQUATIONS

The static and dynamic node-point equilibrium equations for any structural system, with N_d displacement degrees-of-freedom (DOF), can be written in the following general form:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{R}(t) + \mathbf{R}_D(\mathbf{u}, \dot{\mathbf{u}}, t) = \overline{\mathbf{F}}\mathbf{g}(t) \tag{1}$$

At time t the node acceleration, velocity, displacement and external applied load vectors are defined by $\ddot{\mathbf{u}}(t)$, $\dot{\mathbf{u}}(t)$, $\mathbf{u}(t)$ and $\mathbf{R}(t)$, respectively.

The unknown force vectors, $\mathbf{R}_D(\mathbf{u}, \dot{\mathbf{u}}, t)$, are the forces associated with internal energy dissipation such as damping and nonlinear forces. In most cases, these forces are self-equilibrating and do not contribute to the global equilibrium of the total structure [1].

The sum of $\mathbf{R}$ and $\mathbf{R}_D$ can always be represented by the product $\overline{\mathbf{F}}\mathbf{g}(t)$, where $\mathbf{F}$ is an N_d by L matrix of L linearly independent spatial load vectors associated with both linear and nonlinear behavior, and $\mathbf{g}(t)$ is a vector of L time functions. These time functions are directly specified for linear analysis, and are evaluated by iteration for nonlinear elements.

For many problems, nonlinear forces may be restricted to a subset of all DOF, so that $L < N_d$, although this is not required in what follows.

The node-point lumped mass matrix, $\mathbf{M}$, need not have mass associated with all degrees-of-freedom; therefore, it may be singular and mathematically positive semi-definite. Also, external loads may be applied to displacement DOF that do not have mass and produce only static displacements.

The linear elastic stiffness matrix $\mathbf{K}$ may contain rigid-body displacements, as is the case for ship and aerospace structures; therefore, it need not be positive-definite. In order to overcome this potential singularity the term $\rho\,\mathbf{M}\mathbf{u}(t)$ may be added to both sides of the equilibrium equations, where ρ is an arbitrary positive number. Or, Equation (1) can be written as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \overline{\mathbf{K}}\mathbf{u}(t) = \mathbf{F}\mathbf{g}(t) + \rho\,\mathbf{M}\mathbf{u}(t) = \overline{\mathbf{R}}(\mathrm{t}) \tag{2}$$

While $\mathbf{K}$ and $\mathbf{M}$ may be singular, it is assumed here that the effective-stiffness matrix, $\overline{\mathbf{K}} = \mathbf{K} + \rho\,\mathbf{M}$, is nonsingular. Therefore, the effective-stiffness matrix represents a real structure with the addition of external springs to all mass DOF; these springs have stiffness proportional to the mass matrix.

The purpose of this paper is to present a general solution method for the numerical calculation of displacement and member forces. The proposed method can be used for both static and dynamic loads and has the ability to include arbitrary damping and nonlinear energy dissipation. The derivation of the vector-generation algorithm presented in this paper is self-contained and only uses the fundamental laws of physics and mathematics. Near the end of the paper, it will be pointed out that each step in the solution algorithm is nothing more than the application of well-known numerical techniques that have existed for over fifty years. It is an extension of Load Dependent Ritz vectors that have been previously described in [1].

CHANGE OF VARIABLE

Equation (2) is an exact equilibrium statement for the structure at all points in time. The first step in the static or dynamic solution of this fundamental equilibrium equation is to introduce the following change of variable:

$$\mathbf{u}(t) = \mathbf{\Phi}\mathbf{Y}(t) \quad \text{and} \quad \ddot{\mathbf{u}}(t) = \mathbf{\Phi}\ddot{\mathbf{Y}}(t) \tag{3}$$

The N_d by N matrix $\mathbf{\Phi}$ of spatial vectors are calculated and normalized to satisfy the following orthogonality equations:

$$\mathbf{\Phi}^T \mathbf{M}\mathbf{\Phi} = \mathbf{\Psi} \tag{4}$$

$$\mathbf{\Phi}^T \overline{\mathbf{K}}\mathbf{\Phi} = \mathbf{I} \qquad \text{or,} \qquad \mathbf{\Phi}^T \mathbf{K}\mathbf{\Phi} = \mathbf{I} - \rho\mathbf{\Psi} \tag{5}$$

The N by N diagonal matrices are $\mathbf{I}$ for the unit matrix and $\mathbf{\Psi}$ for the generalized mass matrix associated with each vector. Therefore, Equation (2) can be written as a set of uncoupled equations of the following form:

$$\mathbf{\Psi}\ddot{\mathbf{Y}}(t) + \mathbf{I}\mathbf{Y}(t) = \mathbf{\Phi}^{\mathrm{T}}\overline{\mathbf{R}}(t) \tag{6}$$

If N equals N_d , the introduction of this simple change of variable into Equation (2) does not introduce any additional approximations. The number of nonzero terms in the diagonal matrix $\mathbf{\Psi}$ indicates the maximum number of dynamic vectors and is equal to the number of lumped masses in the system (or, mathematically, the rank of the mass matrix). If a vector has zero generalized mass it indicates that it is a static response vector.

It is not practical to calculate all N_d static and dynamic shape functions for a large structure. First, it would require a large amount of computer time and storage. Second, a large number of vectors that are not excited by the loading may be calculated. Therefore, a truncated set of N natural eigenvectors will be calculated that will produce an accurate solution for an ***optimum number of NLD vectors***.

In order to minimize the number of shape functions required to obtain an accurate solution the static displacement vectors produced by L linearly independent spatial functions $\overline{\mathbf{F}}^{(1)}$ associated with the loading $\overline{\mathbf{R}}(t)$ will be used to generate the first set of vectors. The linear independent spatial load functions $\overline{\mathbf{F}}^{(1)}$ can be automatically extracted from $\overline{\mathbf{R}}(t)$ based on the type of external global loading and the location of the nonlinear elements.

CALCULATION OF STIFFNESS ORTHOGONAL VECTORS

The first step in the calculation of the orthogonal vectors defined by Equation (4) and (5) is to calculate a set of stiffness orthogonal vectors $\mathbf{V}$ where each vector satisfies the following equation:

$$\mathbf{v}_{\mathrm{m}}^{\mathrm{T}}\overline{\mathbf{K}}\mathbf{v}_{n} = \mathbf{v}_{\mathrm{m}}^{\mathrm{T}}\mathbf{f}_{n} = \begin{Bmatrix} 1 \text{ for } m = n \\ 0 \text{ for } m \neq n \end{Bmatrix} \tag{7}$$

These stiffness orthogonal displacement and load vectors are calculated and stored in the following arrays:

$$\begin{aligned} \mathbf{V} &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & - & - & \mathbf{v}_N & - & - & - & - & \mathbf{v}_{N_d} \end{bmatrix} \\ \mathbf{F} &= \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 & - & - & \mathbf{f}_N & - & - & - & - & \mathbf{f}_{N_d} \end{bmatrix} \end{aligned} \tag{8}$$

All vectors are generated in sequence $n = 1, 2, \text{----}N$. After each vector is made stiffness orthogonal and normalized it is inserted into position N. For example, consider a new displacement candidate vector $\overline{\mathbf{v}}$ (produced by the load vector $\overline{\mathbf{f}}$) that is not stiffness orthogonal as define by Equation (7). This vector can be modified to be stiffness orthogonal by conducting the following numerical operations:

1. Normalization vector by the application of the following equations:

$$\hat{\mathbf{v}} = \hat{\beta}\ \overline{\mathbf{v}} \quad \text{and} \quad \hat{\mathbf{f}} = \hat{\beta}\ \overline{\mathbf{f}} \quad \text{where} \quad \hat{\beta} = \sqrt{\overline{\mathbf{v}}^{\mathbf{T}}\overline{\mathbf{f}}} \quad ; \text{ therefore } \quad \hat{\mathbf{v}}^{\mathbf{T}}\hat{\mathbf{f}} = \mathbf{1} \tag{9}$$

2. Remove from $\hat{\mathbf{v}}$ all previously calculated stiffness orthogonal vectors by the application of the following equations:

$$\tilde{\mathbf{v}} = \hat{\mathbf{v}} - \sum_{n=1}^{N-1} \alpha_n \mathbf{v}_n \quad \text{and} \quad \tilde{\mathbf{f}} = \hat{\mathbf{f}} - \sum_{n=1}^{N-1} \alpha_n \mathbf{f}_n \qquad \text{(10a and b)}$$

Multiplication of Equation (10a) by $\mathbf{v}_n^T \mathbf{K}$ yields the following equation:

$$\mathbf{v}_n^T \mathbf{K} \tilde{\mathbf{v}} = \mathbf{v}_n^T \mathbf{K} \hat{\mathbf{v}} - \alpha_n \mathbf{v}_n^T \mathbf{K} \mathbf{v}_n \qquad n = 1 \text{ to } N \tag{11}$$

If the new vector $\tilde{\mathbf{v}}$ is to be stiffness orthogonal $\mathbf{v}_n^T \mathbf{K} \tilde{\mathbf{v}}$ must equal zero. Therefore,

$$\alpha_n = \mathbf{v}_n^T \mathbf{K} \hat{\mathbf{v}} \tag{12}$$

3. After Equations (10a) and (10b) are evaluated they must be normalized by the application of the equations:

$$\mathbf{v}_N = \beta\, \tilde{\mathbf{v}} \text{ and } \mathbf{f}_N = \beta\, \tilde{\mathbf{f}} \text{ where } \beta = \sqrt{\tilde{\mathbf{v}}^{\mathbf{T}} \tilde{\mathbf{f}}} \text{ ; therefore } \mathbf{v}_N^T \mathbf{f}_N = \mathbf{1} \tag{13}$$

4. It is now possible to check if the candidate vector $\overline{\mathbf{v}}$ was linearly independent of the previously calculated vectors by checking if the proposed new vector $\mathbf{v}_N$ is nothing more than numerical round-off. Therefore,

$$\text{If } \beta < tol \text{ reject } \mathbf{v_N} \text{ as a new stiffness orthogonal vector} \tag{14}$$

The value for tol is selected to be approximately 10^{-7}.

The first block of candidate vectors is obtained by solving the following set of equations, where the static loads $\overline{\mathbf{F}}^{(1)}$ and displacements $\overline{\mathbf{V}}^{(1)}$ are N_d by L matrices:

$$\overline{\mathbf{K}\mathbf{V}}^{(1)} = \mathbf{LDL^T}\overline{\mathbf{V}}^{(1)} = \overline{\mathbf{F}}^{(1)} \tag{15}$$

Note that the effective stiffness matrix need be triangularized, $\overline{\mathbf{K}} = \mathbf{LDL^T}$, only once. Additional blocks of candidate vectors can be generating from the solution of the following recursive equation:

$$\overline{\mathbf{K}\mathbf{V}}^{(i)} = \mathbf{M}\overline{\mathbf{V}}^{(i-1)} = \overline{\mathbf{F}}^{(i)} \tag{16}$$

If, during the orthogonality calculation, a new displacement or load vector in the block is identified as the same (parallel) as a previously calculated vector it can be discarded from the block and the algorithm is continued with a reduced block size. If the block size is reduced to zero, prior to the production of N_d vectors, it indicates that all of the static and dynamic vectors, excited by the initial load patterns, have been found.

MASS ORTHOGONALITY

After all blocks of the stiffness orthogonal vectors are calculated they can be made orthogonal to the mass matrix by the introduction of the following transformation:

$$\Phi = \mathbf{VZ} \tag{17}$$

Substitution of Equation (17) into Equation (4) produces the following N by N eigenvalue problem:

$$\overline{\mathbf{M}}\mathbf{Z} = \Psi \tag{18}$$

where $\overline{\mathbf{M}} = \mathbf{V}^{\mathrm{T}}\mathbf{MV}$. The stiffness and mass orthogonal vectors are then calculated from Equation (17). The static modes have zero periods, or $\Psi_n = 0$. Therefore, in order to avoid all potential numerical problems, it is recommended that the classical Jacobi rotation method be used to extract the eigenvalues and vectors of this relatively small eigenvalue problem [1].

Equation (2) can now be rewritten as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \overline{\mathbf{K}}\mathbf{u}(t) - \rho\,\mathbf{Mu}(t) = \mathbf{Fg}(t) \tag{19}$$

The transformation to modal coordinates produces the following uncoupled model equations:

$$\boldsymbol{\Psi}\ddot{\mathbf{Y}}(t) + [\mathbf{I} - \rho\,\boldsymbol{\Psi}]\mathbf{Y}(t) = \boldsymbol{\Phi}^T\,\mathbf{Fg}(t) \tag{20}$$

Therefore, a typical modal equation, n, can be written as

$$\Psi_n \ddot{Y}_n(t) + [1 - \rho\,\Psi_n]Y_n(t) = \phi_n^T\,\mathbf{Fg}(t) \tag{21}$$

The number of static shape functions is equal to the number of zero diagonal terms in the matrix $\mathbf{\Psi}$. For the static modes Ψ_n is equal to zero and the solution is written as

$$Y_n(t) = \phi_n^T \mathbf{Fg}(t) \tag{22}$$

For the dynamic elastic modes the generalized mass for each mode is Ψ_n and the classical free-vibration frequencies (radians per second) and the periods of vibrations (seconds) can be calculated from

$$\omega_n = \sqrt{\frac{1}{\Psi_n} - \rho} \quad \text{and} \quad T_n = \frac{2\pi}{\omega_n} \tag{23}$$

Note that the eigenvalue Ψ_n always has a finite numerical value; however, the frequency ω_n and period T_n can have infinite numerical values and cannot be numerically calculated directly for all modes. For example, Table 1 summarizes the eigenvalue, frequencies and periods for the simple beam shown in Figure 1.

Table 1. Eigenvalues for Simple Beam for $\rho = 0.01$

Mode Number	Eigenvalue Ψ_n	Frequency $\omega_n = \sqrt{\frac{1}{\Psi_n} - \rho}$	Period $T_n = \frac{2\pi}{\omega_n}$
1	100	0	∞
2	100	0	∞
3	0.826	0.995	6.31
4	0	∞	0
5	0	∞	0
6	0	∞	0

The generalized stiffness and mass for the normalized vectors are as follows:

$$\phi_n^T \mathbf{K} \phi_n = 1 - \rho \ \Psi_n = \begin{cases} 0 & \text{for rigid - body modes} \\ \omega_n^2 \Psi_n & \text{for dynamic modes} \\ 1 & \text{for static modes} \end{cases} \tag{25}$$

$$\phi_n^T \mathbf{M} \phi_n = \Psi_n = \begin{cases} 1/\rho & \text{for rigid - body modes} \\ \Psi_n & \text{for dynamic modes} \\ 0 & \text{for static modes} \end{cases} \tag{26}$$

Therefore, it is necessary to save both the generalized stiffness and generalized mass, for each mode, in order to determine the static, dynamic or rigid-body response analysis of the mode.

The solution for the dynamic modes can be obtained using the piece-wise exact algorithm [1]. For all rigid-body modes Ψ_n will equal $1/\rho$. Therefore, their response can be calculated by direct, numerical or exact, integration from

$$\ddot{Y}_n(t) = \rho \ \phi_n^T \mathbf{R}(t), \ \dot{Y}_n(t) = \int \ddot{Y}_n(t)\, dt \ , \text{ and } \ Y_n(t) = \int \dot{Y}_n(t) dt \tag{24}$$

The sum of the static, dynamic and rigid-body responses produces a unified method for the static and dynamic analysis of all types of structural systems.

MATHEMATICAL CONSIDERATIONS

Except for reference to the Jacobi and piece-wise integration methods [1], the numerical method for the generation of stiffness and mass orthogonal vectors presented is based on the fundamentals of mechanics and requires no additional references to completely understand. However, it is very interesting to note that the method is nothing more than the application of several well-known numerical techniques:

First, the change of variables introduced by Equation (3) in an application of the standard method of solving differential equation and is also known as the ***separation of variables*** in which the solution the solution is expressed in terms of the product of space functions and time functions. A special application of this approach in classical structural dynamics is called the ***mode superposition method*** in which the static mode response is neglected.

Second, the addition of the term $\rho\,\mathbf{M}\mathbf{u}(t)$ to the stiffness matrix is called an eigenvalue shift in mathematics. However, it is worth noting the zero eigenvalues associated with the static modes are not shifted.

Third, the recurrence relationship, Equation (10), is identical to the inverse iteration algorithm for a single vector [2]. Therefore, the approach is a ***power method*** that will always converge to the lowest eigenvalues of the system.

Fourth, the series of vectors generated by the inverse iteration method is known as the ***Krylov Subspace*** [2]. A. N. Krylov, 1863-1945, was a great Russian engineer and mathematician who first studied the dynamic response of ship structures. However, Krylov did not include static modes in his work.

Fifth, orthogonality is maintained, Equations (14), by the application of the ***modified Gram-Schmidt algorithm***. Theoretically, after the initial block of orthogonal vectors are calculated, it is only necessary to make each new displacement vector orthogonal with respect to the previous two Krylov vectors. However, after many years of experience with the dynamic analysis of very large structural systems, we have found that it is necessary to apply the Gram-Schmidt method to all previously calculated vectors in order that the same vectors are not regenerated.

Sixth, the performance of the algorithm is improved if the load vectors $\mathbf{F}^{(i)}$, for each block, are made orthogonal with respect to the previously calculated displacement vectors, Equation (16), prior to the solution of the equilibrium equations. This additional step has made the algorithm ***unique and very robust***.

Finally, an appropriate name for the new algorithm to generate a truncated set of static and dynamic **N**atural **L**oad **D**ependent vectors is the **NLD** vector algorithm.

LOAD PARTICIPATION RATIOS and ERROR ESTIMATION

In the analysis of structures subjected to three base accelerations there is a requirement that one must include enough modes to account for 90 percent of the mass in the three global directions. However, for other types of loading, such as base displacement loads, there are no guidelines as to how many modes are to be used in the analysis. The purpose of this section is to define two new load participation ratios, which can be calculated during the generation of the NLD vectors, to assure that an adequate number of vectors are used in a subsequent static or dynamic analysis.

From Equation (24), a typical modal equation n for load pattern j, can be written as

$$\Psi_n \ddot{Y}(t)_n + \omega_n^2 \Psi_n Y(t)_n = \phi_n^T \mathbf{F}_j g(t)_j \qquad n = 1 \text{ to } N \tag{27}$$

The error indicators are based on the two different types of load functions $g(t)_j$. In one case the loads vs. time excite the low frequencies; and, in the other case the high frequencies are excited.

Static Loads

The first error estimator is a measure of the ability of a truncated set of mode shapes to capture the static response of the structural system. For this case the load function $g(t)_j$ is applied linearly from a value of zero at time zero to a value of 1.0 at the end of a very large time interval. Therefore, the inertia terms can be neglected and Equation (27), evaluated at the end of the large time interval, is

$$\omega_n^2 \Psi_n Y_{ns} = \phi_n^T \mathbf{F}_j \qquad n = 1 \text{ to } N \tag{28}$$

Therefore the ***static mode participation*** can be written as

$$Y_{nj} = \frac{\phi_n^T \mathbf{F}_j}{\omega_n^2 \Psi_n} \quad n = 1 \text{ to } N \tag{29}$$

From Equation (3) the approximate static displacement response of the structure due to N modes is

$$\overline{\mathbf{u}}_j = \sum_{n=1}^{N} \phi_n Y_{nj} \tag{30}$$

The approximate strain energy associate with the displacement defined by Equation (30) is

$$\overline{E}_{sj} = \frac{1}{2}\overline{\mathbf{u}}_j^T \mathbf{K}\,\overline{\mathbf{u}}_j = \frac{1}{2}\sum_{n=1}^{N} Y_{nj}\phi_n^T \mathbf{K}\phi_n Y_{nj} = \frac{1}{2}\sum_{n=1}^{N}\omega_n^2 \Psi_n \phi_n Y_{nj}^2 = \frac{1}{2}\sum_{n=1}^{N}\frac{(\phi_n^T \mathbf{F}_j)^2}{\omega_n^2 \Psi_n} \tag{31}$$

The exact static displacement due to the load pattern can be calculated from the solution of the following static equilibrium equation:

$$\mathbf{K}\,\mathbf{u}_j = \mathbf{F}_j \tag{32}$$

The exact strain energy stored in the structure for the load pattern is calculated from

$$E_{sj} = \frac{1}{2}\mathbf{u}_j^T \mathbf{K}\,\mathbf{u}_j = \frac{1}{2}\mathbf{u}_j^T \mathbf{F}_j \tag{33}$$

The ***static load participation ratio*** is defined as the ratio of the strain energy captured by the truncated set of vectors, $\overline{E}_j$, to the total strain energy, E_j. For the typical case where $\rho = 0$ the ratio is

$$r_{sj} = \frac{\sum_{n=1}^{N}(\phi_n^T \mathbf{F}_j)^2}{\mathbf{u}_j^T \mathbf{F}_j} \tag{34}$$

It must be pointed out that for NLD vectors, this ratio is always equal to 1.0. Whereas, the use of the exact dynamic eigenvectors may require a large number of vectors in order to capture the static load response. Also, if the static mode shapes are excited it is not possible for the exact eigenvectors to converge to the exact static solution.

Dynamic Response

The ***dynamic load participation ratio*** is based on the use of the application of the static loads as a delta function at time zero that produces an initial condition for a free vibration response analysis of the total structural system. It is well known that any type of time function can be represented by the sum of these impulse functions applied at different points in time. This type of loading will produce an initial velocity at the mass points of $\dot{\mathbf{u}}_j = \mathbf{M}^{-1}\mathbf{F}_j$. Therefore, the total kinetic input to the system, for a typical load vector *j*, is given by

$$E_{kj} = \frac{1}{2}\dot{\mathbf{u}}_j^T\mathbf{M}\dot{\mathbf{u}}_j = \frac{1}{2}\mathbf{f}_j^T\mathbf{M}^{-1}\mathbf{F}_j \tag{35}$$

From Equation (3) the relationship between initial node velocities and the initial modal velocities is

$$\dot{\mathbf{u}}_j = \sum_{\mathrm{n}=1}^{\mathrm{N}} \phi_{\mathrm{n}} \dot{Y}_{nj} \tag{36}$$

Therefore, the kinetic energy associated with the truncated set of vectors is

$$\overline{E}_{kj} = \frac{1}{2}\dot{\mathbf{u}}_j^T\mathbf{M}\dot{\mathbf{u}}_j = \frac{1}{2}\sum_{n=1}^{N}\Psi_n \dot{Y}_{nj}^2 \tag{37}$$

The initial modal velocity $\dot{Y}_{nj}$ is obtained from the solution of Equation (27) as

$$\dot{Y}_{nj} = \frac{\varphi_n^T\mathbf{F}_j}{\Psi_n} \tag{38}$$

Substitution of Equation (38) into Equation (37) yields

$$\overline{E}_{jk} = \frac{1}{2}\sum_{n=1}^{N}\frac{(\varphi_n^T\mathbf{F}_j)^2}{\Psi_n} \tag{39}$$

The ***dynamic load participation ratio*** is defined as the ratio of the kinetic energy captured by the truncated set of vectors, $\overline{E}_{kj}$, to the total kinetic energy, E_{kj}. For the typical case where $\rho = 0$ the ratio is

$$r_{dj} = \frac{\sum_{n=1}^{N}(\omega_n \phi_n^T \mathbf{F}_j)^2}{\mathbf{F}_j^T\mathbf{M}^{-1}\mathbf{F}_j} \tag{40}$$

A dynamic load participation ratio equal to 1.0 assures that all the energy input is captured for the dynamic load condition *j*. In the case of base acceleration loading where the three load vectors are the directional masses the dynamic load participation ratios are identical to the mass participation ratios.

Automatic Termination of NLD Vectors

Since the NLD vector algorithm starts with a full set of static vectors the static load participation factor will always equal 1.0. Equation (40), the dynamic load participation factor can be evaluated after each block of vectors are generated. Therefore, this factor can be computed as the vectors are calculated and it can be used as an indicator to automatically terminate the generation of NLD vectors. Based on experience, a dynamic load participation ratio of at least 0.95, for all load patterns, will assure accurate results for most types of loading. This is a very important user option since the number of vectors requested need not be specified prior to the dynamic analysis.

USE OF THE NLD ALGORITHM TO CALCULATE EIGENVECTORS

The NLD vector algorithm, as presented in this paper, generates the complete Krylov subspace for a specified set of load vectors and errors in the resulting dynamic response analysis are minimized. If one examines the frequencies associated with the vectors one finds that all of the lower frequencies are identical to the frequencies obtained from an exact eigenvalue analysis. Since the approach is related to the power method this is to be expected. The higher modes produced by the NLD vector algorithm are linear combinations of the exact eigenvectors and components of the static response vectors. It is the optimum set of vectors to solve the dynamic response problem associated with the specified static load patterns. Therefore, the number of NLD vectors required will always be less than if the exact eigenvectors were used.

If, for some reason, one wishes to calculate the exact eigenvalues and vectors the same numerical method can be used. The initial displacement vectors need only be set to random vectors. If, during the generation, vectors are generated which are identical to previously calculated vectors they can be replaced with new random displacement vectors. The procedure can be terminated at any time; however, the higher frequencies will not be exact. The introduction of iteration for each block can be used to used to calculate the exact eigenvalues and vectors. Note that if the system contains M masses, the method will generate M exact eigenvectors; nevertheless, if random load vectors are used directly, instead of $\overline{\mathbf{F}}^{(i)} = \mathbf{M}\mathbf{V}^{(i-1)}$, the algorithm can continue and will produce N_d-M static response vectors which have infinite frequencies and zero periods.

SUMMARY

The use of exact eigenvectors to reduce the number of degrees of freedom required to conduct a dynamic response analysis has significant limitations. The effects of the application of static loads to massless DOF cannot be taken into account. In addition, for certain types of loading a large number of vectors are required. On the other hand, a large number of exact eigenvectors may be calculated that are not excited by the loading on the structure.

The use of static and dynamic Natural Load Dependent, NLD, vectors, presented in this paper, eliminate the problems associated with the use of the exact eigenvectors. In addition, the NLD vector algorithm produces a unified approach to the static and dynamic analysis of many different types of structural systems. In addition, it is possible to check if an adequate number of vectors are generated prior to the integration of the equations of motion.

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