2.

EQUILIBRIUM AND COMPATIBILITY

Equilibrium Is Essential - Compatibility Is Optional

2.1 INTRODUCTION

Equilibrium equations, which set the externally applied loads equal to the sum of the internal element forces at all joints, or node points, of a structural system, are the most fundamental equations in structural analysis and design. The exact solution for a problem in solid mechanics requires that the differential equations of equilibrium for all infinitesimal elements within the solid must be satisfied. *Equilibrium is a fundamental law of physics and cannot be violated within a "real" structural system.* Therefore, it is critical that the mathematical model, which is used to simulate the behavior of a real structure, also satisfies these basic equilibrium equations.

It is important to note that within a finite element, which is based on a formal displacement formulation, the differential stress-equilibrium equations are not always satisfied. However, inter-element force-equilibrium equations are identically satisfied at all node points (joints). The computer program user, who does not understand the approximations used to develop a finite element, can obtain results that are in significant error if the element mesh is not sufficiently fine in areas of stress concentration [1].

Compatibility requirements should be satisfied; however, if one has a choice between satisfying equilibrium or compatibility one should use the equilibrium based solution. For real nonlinear structures, equilibrium is always satisfied in the deformed position. Many real structures do not satisfy compatibility due to creep, joint slippage, incremental construction and directional yielding.

2.2 FUNDAMENTAL EQUILIBRIUM EQUATIONS

The three-dimensional equilibrium of an infinitesimal element, shown in Figure 1.1, is given by the following equilibrium equations [2]:

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + \beta_1 = 0$$

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \tau_{13,3} + \beta_1 = 0 \tag{2.1}$$

$$\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + \beta_1 = 0$$

The body force, β_i, is per unit of volume in the i-direction and represents gravitational forces or pore pressure gradients. Since $\tau_{ij} = \tau_{ji}$ the infinitesimal element is automatically in rotational equilibrium. Of course, for this equation to be valid for large displacements it must be satisfied in the deformed position and all stresses must be defined as force per unit of deformed area.

2.3 STRESS RESULTANTS - FORCES AND MOMENTS

In structural analysis it is standard practice to write equilibrium equations in terms of stress resultants rather than in terms of stresses. Force stress resultants are calculated by the integration of normal or shear stresses acting on a surface.

Moment stress resultants are the integration of stresses on a surface times a distance from an axis.

A point load, which is a stress resultant, is by definition an infinite stress times an infinitesimal area and is physically impossible on all real structures. Also, a point moment is a mathematical definition and does not have a unique stress field as a physical interpretation. Clearly, the use of forces and moments is fundamental in structural analysis and design. However, a clear understanding of their use in finite element analysis is absolutely necessary if stress results are to be physically evaluated.

For a finite size element or joint, a substructure, or a complete structural system *the following six equilibrium equations must be satisfied:*

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \tag{2.2}$$

For two dimensional structures only three of these equations need be satisfied.

2.4 COMPATIBILITY REQUIREMENTS

For continuous solids we have defined strains as displacements per unit length. In order to calculate absolute displacements at a point we must integrate the strains with respect to a fixed boundary condition. This integration can be conducted over many different paths. A solution is compatible if the displacement at all points is not a function of the path. Therefore, a displacement compatible solution involves the existence of a uniquely defined displacement field.

In the analysis of a structural system of discrete elements, all elements connected to a joint, or node point, must have the same absolute displacement. If the node displacements are given, all element deformations can be calculated from the basic equations of geometry. In a displacement based finite element analysis, node displacement compatibility is satisfied. However, it is not necessary that the displacements along the sides of the elements are compatible if the element passes the "patch test".

A finite element passes the patch test "if a group (or patch) of elements, of arbitrary shape, is subjected to node displacements associated with constant strain; and, the results of a finite element analysis of the patch of elements yield constant strain". In the case of plate bending elements, the application of a constant curvature displacement pattern at the nodes must produce constant curvature within a patch of elements. If an element does not pass the patch test it may not converge to the exact solution. Also, in the case of a coarse mesh, elements that do not pass the patch test may produce results with significant errors.

2.5 STRAIN-DISPLACEMENT EQUATIONS

If the small displacement fields u_1, u_2 and u_3 are specified, assumed or calculated, the consistent strains can be calculated directly from the following well-known strain-displacement equations [2]:

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} \tag{2.3a}$$

$$\varepsilon_2 = \frac{\partial u_2}{\partial x_2} \tag{2.3b}$$

$$\varepsilon_3 = \frac{\partial u_3}{\partial x_3} \tag{2.3c}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \tag{2.3d}$$

$$\gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \tag{2.3e}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \tag{2.3f}$$

2.6 DEFINITION OF ROTATION

A unique rotation at a point in a real structure does not exist. A rotation of a horizontal line may be different from the rotation of a vertical line. However, in many theoretical books on continuum mechanics the following mathematical equations are used to define rotation of the three axes:

$$\theta_3 \equiv \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right] \tag{2.4a}$$

$$\theta_2 \equiv \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right] \tag{2.4b}$$

$$\theta_1 \equiv \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right] \tag{2.4c}$$

It is of interest to note that this definition of rotation is the average rotation of two normal lines. It is important to recognize that these definitions are not the same as used in beam theory when shearing deformations are included. When beam sections are connected the absolute rotation of the end sections must be equal.

2.7 EQUATIONS AT MATERIAL INTERFACES

One can clearly understand the fundamental equilibrium and compatibility requirements from an examination of the stresses and strains at the interface between two materials. A typical interface, for a two dimensional continuum, is shown in Figure 2.1. By definition, the displacements at the interface are equal. Or, $u_s(s,n) = \overline{u}_s(s,n)$ and $u_n(s,n) = \overline{u}_n(s,n)$.

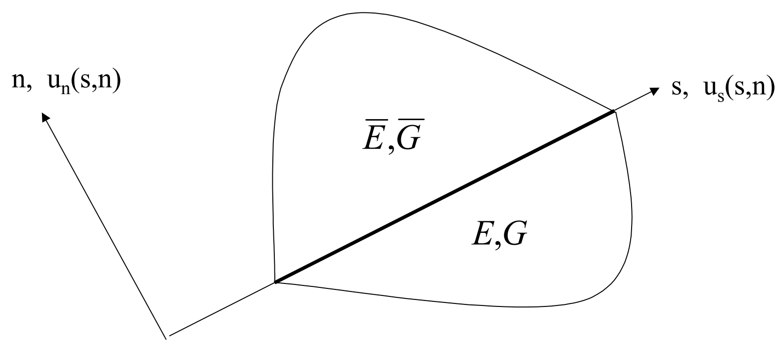

Figure 2.1. Material Interface Properties

Normal equilibrium at the interface requires that the normal stresses are equal. Or,

$$\sigma_n = \overline{\sigma}_n \tag{2.5a}$$

Also, the shear stresses at the interface are equal. Or,

$$\tau_{ns} = \overline{\tau}_{ns} \tag{2.5b}$$

Since the displacement, u_s and $\overline{u}_s$, must be equal and continuous at the interface

$$\varepsilon_s = \overline{\varepsilon}_s \tag{2.5c}$$

Since the material properties, that relate stress to strain, are not equal for the two materials it can be concluded that

$$\sigma_s \neq \overline{\sigma}_s \tag{2.5d}$$

$$\varepsilon_n \neq \overline{\varepsilon}_n \tag{2.5e}$$

$$\gamma_{ns} \neq \overline{\gamma}_{ns} \tag{2.5f}$$

For a three dimensional material interface, on a s-t surface, it is apparent that the following 12 equilibrium and compatibility equations exist:

$$\sigma_n = \overline{\sigma}_n \qquad \varepsilon_n \neq \overline{\varepsilon}_n \tag{2.6a}$$

$$\sigma_s \neq \overline{\sigma}_s \qquad \varepsilon_s = \overline{\varepsilon}_s \tag{2.6b}$$

$$\sigma_t \neq \overline{\sigma}_t \qquad \varepsilon_t = \overline{\varepsilon}_t \tag{2.6c}$$

$$\tau_{ns} = \overline{\tau}_{ns} \qquad \gamma_{ns} \neq \overline{\gamma}_{ns} \tag{2.6d}$$

$$\tau_{nt} = \overline{\tau}_{nt} \qquad \gamma_{nt} \neq \overline{\gamma}_{nt} \tag{2.6e}$$

$$\tau_{st} \neq \overline{\tau}_{st} \qquad \gamma_{st} = \overline{\gamma}_{st} \tag{2.6f}$$

These 12 equations cannot be derived because they are fundamental physical laws of equilibrium and compatibility. It is important to note that if a stress is continuous the corresponding strain, derivative of the displacement, is discontinuous. Also, if a stress is discontinuous the corresponding strain, derivative of the displacement, is continuous.

The continuity of displacements between elements and at material interfaces is defined as C_0 displacement fields. Elements with continuities of the derivatives of the displacements are defined by C_1 continuous elements. It is apparent that elements with C_1 displacement compatibility cannot be used at material interfaces.

2.8 INTERFACE EQUATIONS IN FINITE ELEMENT SYSTEMS

In the case of a finite element system in which the equilibrium and compatibility equations are satisfied only at node points along the interface, the fundamental equilibrium equations can be written as

$$\sum F_n + \sum \overline{F}_n = 0 \tag{2.7a}$$

$$\sum F_s + \sum \overline{F}_s = 0 \tag{2.7b}$$

$$\sum F_t + \sum \overline{F}_t = 0 \tag{2.7c}$$

Each node on the interface between elements has a unique set of displacements; therefore, compatibility at the interface is satisfied at a finite number of points. As the finite element mesh is refined the element stresses and strains approach the equilibrium and compatibility requirements given by Equations (2.6). Therefore, each element in the structure may have different material properties.

2.9 STATICALLY DETERMINATE STRUCTURES

The internal forces of some structures can be determined directly from the equations of equilibrium only. For example, the truss structure shown in Figure 2.2 will be analyzed in order to illustrate that the classical "method of joints" is nothing more than solving a set of equilibrium equations.

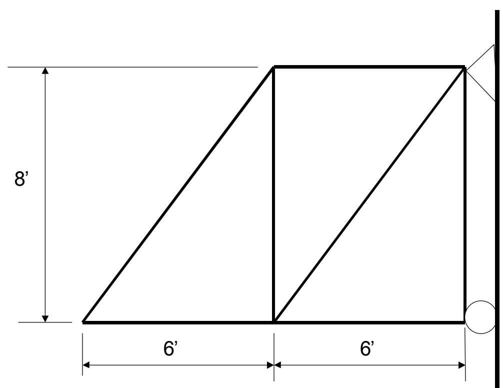

Figure 2.2. Simple Truss Structure

Positive external node loads and node displacements are shown in Figure 2.3. Member forces f_i and deformations d_i are positive in tension.

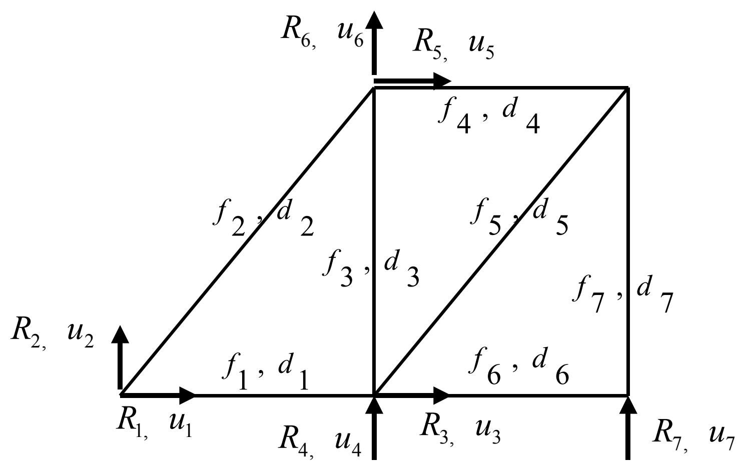

Figure 2.3 Definition of Positive Joint Forces and Node Displacements

Equating two external loads, R_j, at each joint to the sum of the internal member forces, f_i, (see Appendix B for details) yields the following seven equilibrium equations written as one matrix equation:

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \end{bmatrix} = \begin{bmatrix} -1.0 & -0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.8 & 0 & 0 & 0 & 0 & 0 \\ 1.0 & 0 & 0 & 0 & -0.6 & 0 & 0 \\ 0 & 0 & -1.0 & 0 & -0.8 & -1.0 & 0 \\ 0 & 0.6 & 0 & -1.0 & 0 & 0 & 0 \\ 0 & 0.8 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} \tag{2.8}$$

Or, symbolically

$$\mathbf{R} = \mathbf{A}\mathbf{f} \tag{2.9}$$

where $\mathbf{A}$ is a load-force transformation matrix and is a function of the geometry of the structure only. For this ***statically determinate*** structure we have seven unknown element forces and seven joint equilibrium equations; therefore, the above set of equations can be solved directly for any number of joint load conditions. If the structure had one additional diagonal member there would be eight unknown member forces and a direct solution would not be possible because the structure would be ***statically indeterminate***. The major purpose of this example is to express the well-known traditional method of analysis (***method of joints***) in matrix notation.

2.10 DISPLACEMENT TRANSFORMATION MATRIX

After the member forces are calculated there are many different traditional methods to calculate joint displacements. Again, in order to illustrate the use of matrix notation the member deformations d_i will be expressed in terms of joint displacements u_j. Consider a typical truss element shown in Figure 2.4.

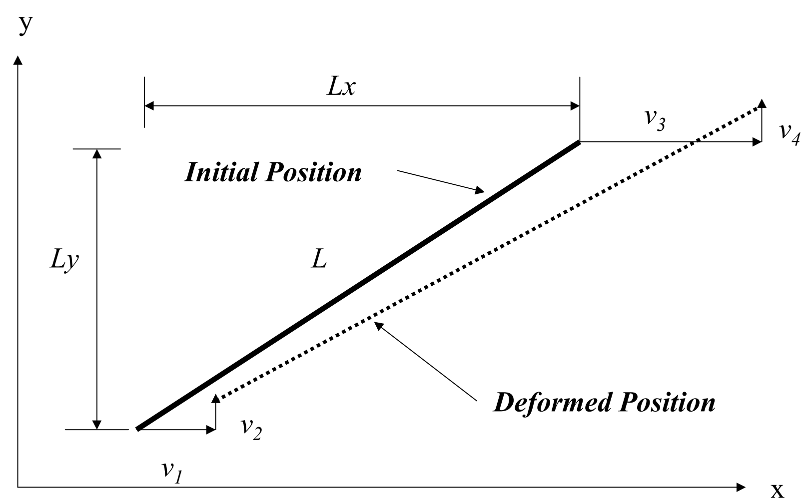

Figure 2.4. Typical Two-Dimension Truss Element

The axial deformation of the element can be expressed as the sum of the axial deformations, due to the four displacements at the two ends of the element. The total axial deformation, written in matrix form, is

$$d = \begin{bmatrix} -\frac{L_x}{L} & -\frac{L_y}{L} & \frac{L_x}{L} & \frac{L_y}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{2.10}$$

Application of Equation (2.10) to all members of the truss, shown in Figure 2.3, yields the following matrix equation:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{bmatrix} = \begin{bmatrix} -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 \\ -0.6 & -0.8 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & -1.0 & 0 & 0 \\ 0 & 0 & -0.6 & -0.8 & 0 & 0 & 0 \\ 0 & 0 & -1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} \tag{2.11}$$

Or, symbolically

$$\mathbf{d} = \mathbf{B}\,\mathbf{u} \tag{2.12}$$

The element deformation-displacement transformation matrix, **B,** is a function of the geometry of the structure. Of greater significance, however, is the fact that the matrix **B** is the transpose of the matrix **A** defined by the joint equilibrium Equation (2.8). Therefore, given the element deformations within this statically determinate truss structure, we can solve Equation (2.11) for the joint displacements.

2.11 ELEMENT STIFFNESS AND FLEXIBILITY MATRICES

The forces in the elements can be expressed in terms of the deformations in the elements by the following matrix equations:

$$\mathbf{f} = \mathbf{k}\,\mathbf{d} \qquad \text{or,}\ \mathbf{d} = \mathbf{k}^{-1}\,\mathbf{f} \tag{2.13}$$

The element stiffness matrix $\mathbf{k}$ is diagonal, for this truss structure, where the diagonal terms are $k_{ii} = \frac{A_i E_i}{L_i}$ and all other terms are zero. The element flexibility matrix is the inverse of the stiffness matrix where the diagonal terms are $\frac{L_i}{A_i E_i}$. It is important to note that the element stiffness and flexibility matrices are only a function of the mechanical properties of the elements.

2.12 SOLUTION OF STATICALLY DETERMINATE SYSTEM

The three fundamental equations of structural analysis for this simple truss structure are equilibrium, Equation (2.8), compatibility, Equation (2.11), and force-deformation, Equation (2.13). For each load condition R, the solution steps can be summarized as follows:

1. Calculate the element forces from Equation (2.8).
2. Calculate element deformations from Equation (2.13).
3. Solve for joint displacements using Equation (2.11).

All traditional methods of structural analysis use these basic equations. However, prior to the availability of inexpensive digital computers, which can solve over 100 equations in less than one second, many special techniques were developed to minimize the number of hand calculations. Therefore, at this point in time, there is little value to summarize these methods in this book on the static and dynamic analysis of structures.

2.13 GENERAL SOLUTION OF STRUCTURAL SYSTEMS

In structural analysis using digital computers, the same equations used in classical structural analysis are applied. The starting point is always joint equilibrium. Or, $\mathbf{R} = \mathbf{A}\,\mathbf{f}$. From the element force-deformation equation, $\mathbf{f} = \mathbf{k}\,\mathbf{d}$, the joint equilibrium equation can be written as $\mathbf{R} = \mathbf{A}\,\mathbf{k}\,\mathbf{d}$. From the compatibility equation, $\mathbf{d} = \mathbf{B}\,\mathbf{u}$, joint equilibrium can be written in terms of joint displacements as $\mathbf{R} = \mathbf{A}\,\mathbf{k}\,\mathbf{B}\,\mathbf{u}$. Therefore, the general joint equilibrium can be written as

$$\mathbf{R} = \mathbf{K}\,\mathbf{u} \tag{2.14}$$

The global stiffness matrix **K** is given by one of the following matrix equations:

$$\mathbf{K} = \mathbf{A}\,\mathbf{k}\,\mathbf{B}\,,\ \text{or,}\ \mathbf{K} = \mathbf{A}\,\mathbf{k}\,\mathbf{A}^{\mathbf{T}}\,,\ \text{or,}\ \mathbf{K} = \mathbf{B}^{\mathbf{T}}\,\mathbf{k}\,\mathbf{B} \tag{2.15}$$

It is of interest to note that the equations of equilibrium or the equations of compatibility can be used to calculate the global stiffness matrix **K**.

The standard approach is to solve Equation (2.14) for the joint displacements and then calculate the member forces from

$$\mathbf{f} = \mathbf{k}\,\mathbf{B}\,\mathbf{u}\,,\ \text{or,}\ \mathbf{f} = \mathbf{k}\,\mathbf{A}^{\mathbf{T}}\,\mathbf{u} \tag{2.16}$$

It should be noted that, within a computer program, the sparse matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{k}$ and $\mathbf{K}$ are never formed because of their large storage requirements. The symmetric global stiffness matrix **K** is formed and solved in condensed form.

2.14 SUMMARY

Internal member forces and stresses must be in equilibrium with the applied loads and displacements. All real structures satisfy this fundamental law of physics. Hence, our computer models must satisfy the same law.

At material interfaces all stresses and strains are not continuous. Computer programs that average node stresses at material interfaces produce plot stress contours that are continuous; however, the results will not converge and significant errors can be introduced by this approximation.

Compatibility conditions, which require that all elements attached to a rigid joint have the same displacement, are fundamental requirements in structural analysis and can be physically understood. Satisfying displacement compatibility involves the use of simple equations of geometry. However, the compatibility equations have many forms and most engineering students and many practicing engineers can have difficulty in understanding the displacement compatibility requirement. Some of the reasons we have difficulty in the enforcement of the compatibility equations are the following:

1. The displacements that exist in most linear structural systems are small compared to the dimensions of the structure. Therefore, deflected shape drawing must be grossly exaggerated in order to write equations of geometry.

2. For structural systems that are statically determinate the internal member forces and stresses can be calculated exactly without the use of the compatibility equations.

3. Many popular (approximate) methods of analysis exist which do not satisfy the displacement compatibility equations. For example, for rectangular frames both the cantilever and portal methods of analysis assume the inflection points to exist at a predetermined location within the beams or columns; therefore, the displacement compatibility equations are not satisfied.

4. Many materials, such as soils and fluids, do not satisfy the compatibility equations. Also, locked in construction stresses, creep and slippage within joints are real violations of displacement compatibility. Therefore,

approximate methods, which satisfy statics, may produce more realistic results for the purpose of design.

5. In addition, engineering students are not normally required to take a course in geometry; whereas, all students take a course in statics. Hence, there has not been an emphasis on the application of the equations of geometry.

The relaxation of the displacement compatibility requirement has been justified for hand calculation in order to minimize computational time. Also, if one must make a choice between satisfying the equations of statics or the equations of geometry, in general, we should satisfy the equations of statics for the reasons previously stated.

However, due to the existence of inexpensive powerful computers and efficient modern computer programs it is not necessary to approximate the compatibility requirements. For many structures, such approximations can produce significant errors in the force distribution in the structure in addition to incorrect displacements.

2.15 REFERENCES

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