APPENDIX G

NUMERICAL INTEGRATION

Exact Integration of Approximate Solutions May Not Produce the Most Realistic Results

G.1 INTRODUCTION

{ XE "Numerical Integration Rules" }Traditional mathematics education implies that exact integration should be used whenever possible. In fact, approximate numerical integration is only recommended in cases where exact integration is not possible. However, in the development of finite element stiffness matrices, which are based on approximate displacement functions that do not satisfy equilibrium, it has been found that approximate numerical integration methods can produce more accurate results, and converge faster, than exact integration.

{ XE "Numerical Integration Rules:5 Point 2D Rule" }{ XE "Numerical Integration Rules:8 Point 2D Rule" }{ XE "Quadrature Rules" }In this appendix, one-, two- and three-dimensional numerical integration formulas will be developed and summarized. These formulas are often referred to as **numerical quadrature rules**. The term **reduced integration** implies that a lower order integration formula is used and certain functions are intentionally neglected. In order that the integration rules are general, the functions to be integrated must be in the range –1.0 to +1.0. A simple change of variable can be introduced to transform any integral to this natural reference system. For example, consider the following one-dimensional integral:

$$I = \int_{x_1}^{x_2} f(x)\, dx \tag{G.1}$$

The introduction of the change of variable $x = \frac{1}{2}(1-r)x_1 + \frac{1}{2}(1+r)x_2$ allows the integral to be written as:

$$I = J\int_{-1}^{-1} f(r)\, dr = JI_r \tag{G.2}$$

It is apparent that:

$$dx = (x_2 - x_1)\, dr = J\,dr \tag{G.3}$$

The mathematical term J is defined as the **Jacobian of the transformation**. For two- and three-dimensional integrals, the Jacobian is more complicated and is proportional to the area and volume of the element respectively. Normally the displacement approximation is written directly in the three-dimensional isoparametric reference system r, s and t. Therefore, no change of variable is required for the function to be integrated.

G.2 ONE-DIMENSIONAL GAUSS QUADRATURE

{ XE "Quadrature Rules" }The integration of a one-dimensional function requires that the integral be written in the following form:

$$I_r = \int_{-1}^{-1} f(r)\, dr = \sum_{i=1}^{N} w_i\, f(r_i) = w_1 f(r_1) + w_2 f(r_2) + ...w_N\, f(r_N) \tag{G.4}$$

The integral is evaluated at the Gauss points r_i and the corresponding Gauss weighting factors are w_i. To preserve symmetry, the Gauss points are located at the center or in pairs at equal location from the center with equal weights.

Let us consider the case where the function to be integrated is a polynomial of the form $f(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + ... a_n r^n$. Or, at a typical numerical integration point:

$$f(r_i) = a_0 + a_1 r_i + a_2 r_i^2 + a_3 r_i^3 + ... a_n r_i^n \tag{G.5}$$

It is apparent that the integrals of the odd powers of the polynomial are zero. The exact integration of the even powers of the polynomial produce the following equation:

$$I_r = \int_{-1}^{-1} f(r)\, dr = \sum_n \int_{-1}^{1} a_n r^n dr = \sum_n \frac{2a_n}{n+1} = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + ... \tag{G.6}$$

A one to three point rule is written as:

$$I_r = w_\alpha f(-\alpha) + w_0 f(0) + w_\alpha f(\alpha) \tag{G.7}$$

Hence, from Equations (G.5) and (G.7), a one point integration rule at $r = 0$ is:

$$I_r = w_0 a_0 = 2a_0 \quad \text{or,} \quad w_0 = 2 \tag{G.8}$$

Similarly, a two-point integration rule at $r = \pm\alpha$ produces:

$$I_r = w_\alpha (a_0 + a_1\alpha + a_2\alpha^2) + w_\alpha (a_0 - a_1\alpha + a_2\alpha^2) = 2a_0 + \frac{2}{3}a_2 \tag{G.9}$$

Equating the coefficients of a_0 and a_2 produces the following equations:

$$\begin{aligned} 2w_\alpha a_0 &= 2a_0 \qquad \text{or,} \quad w_\alpha = 1 \\ 2w_\alpha a_2 \alpha^2 &= \frac{2}{3}a_2 \quad \text{or,} \quad \alpha = \sqrt{\frac{1}{3}} \end{aligned} \tag{G.10}$$

A three-point integration rule requires that:

$$\begin{aligned} I_r = {} & w_\alpha (a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + a_4\alpha^4) + w_0 a_0 \\ & + w_\alpha (a_0 - a_1\alpha + a_2\alpha^2 - a_3\alpha^3 + a_4\alpha^4) = 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 \end{aligned} \tag{G.11}$$

Equating the coefficients of a_0 and a_2 produces the following equations:

$$
\begin{aligned}
&2w_\alpha a_0 + w_0 a_0 = 2a_0 && \text{or,} \quad 2w_\alpha + w_0 = 2 \\
&2w_\alpha a_2 \alpha^2 = \frac{2}{3} a_2 && \text{or,} \quad \alpha^2 = \frac{1}{3w_\alpha} \\
&2w_\alpha a_4 \alpha^4 = \frac{2}{5} a_4 && \text{or,} \quad \alpha^4 = \frac{1}{5w_\alpha}
\end{aligned}
\tag{G.12}
$$

The solution of these three equations requires that:

$$
w_\alpha = \frac{5}{9}, \quad w_0 = \frac{8}{9} \quad \text{and} \quad \alpha = \sqrt{\frac{3}{5}} \tag{G.13}
$$

Note that the sum of the weighting functions for all one-dimensional integration rules are equal to 2.0, or the length of the integration interval from –1 to +1. Clearly one can develop higher order integration rules using the same approach with more integration points. It is apparent that the Gauss method using N points will exactly integrate polynomials of order 2N-1 or less. However, finite element functions are not polynomials in the global reference system if the element is not a rectangle. Therefore, for arbitrary isoparametric elements, all functions are approximately evaluated.

G.3 NUMERICAL INTEGRATION IN TWO DIMENSIONS

The one-dimensional Gauss approach can be extended to the evaluation of two-dimensional integrals of the following form:

$$
I_{rs} = \int_{-1}^{1}\int_{-1}^{1} f(r,s)\, dr\, ds = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j f(r_i s_j) \tag{G.14}
$$

Using one-dimensional Gauss rules in both the r and s directions, Equation (G.14) can be evaluated directly. Two by two integration will require four points and three by three integration requires nine points. For two dimensions, the sum of the weighting factors $w_i w_j$ will be 4.0 or, the area of the element in the natural reference system.

G.4 AN EIGHT-POINT TWO-DIMENSIONAL RULE

It is possible to develop integration rules for two-dimensional elements that produce the same accuracy as the one-dimensional Gauss rules using fewer points. A general, two-dimensional polynomial is of the following form:

$$f(r,s)=\sum_{n,m} a_{nm} r^n s^m \tag{G.15}$$

A typical term in Equation (G.15) may be integrated exactly. Or:

$$\int_{-1}^{1}\int_{-1}^{1} a_{nm} r^n s^m \; dr\, ds = \frac{4\, a_{nm}}{(n+1)(m+1)} \quad \text{if } n \text{ and } m \text{ are both even.} \tag{G.16}$$

A two-dimensional N point integration rule can be written as:

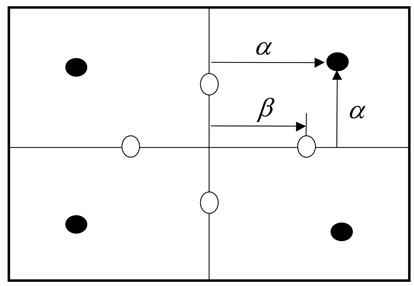
$$\begin{aligned} I = \sum_{i=1}^{N} w_i \; f(r_i, s_i) &= a_{00} \sum_i w_i + a_{10} \sum_i w_i r_i + a_{01} \sum_i w_i s_i \\ &+ a_{11} \sum_i w_i s_i r_i + a_{20} \sum_i w_i r_i^2 + \text{.........} a_{nm} \sum_i w_i r_i^n s_i^m \end{aligned} \tag{G.17}$$

The eight integration points, shown in Figure G.1, produce a two-dimensional rule that can be summarized as:

$$I = w_\alpha \; f(\pm\alpha, \pm\alpha) + w_\beta \, [f(\pm\beta, 0) + f(0, \pm\beta)] \tag{G.18}$$

Equating all non-zero terms in the integrated polynomial of the fifth order produces the following four equations in terms of four unknowns:

$$\begin{aligned} a_{00}:&\quad 4w_\alpha + 4w_\beta = 4 \\ a_{02}\quad a_{20}:&\quad 4w_\alpha \alpha^2 + 2w_\beta \beta^2 = 4/\,3 \\ a_{22}:&\quad 4w_\alpha \alpha^4 = 4/\,9 \\ a_{40}\quad a_{04}:&\quad 4w_\alpha \alpha^4 + 2w_\beta \beta^4 = 4/\,5 \end{aligned} \tag{G.19}$$



$$W_\beta = \frac{40}{49} = ?$$
$$W_\alpha = 1.0 - W_\beta$$
$$\alpha = \sqrt{\frac{1.0}{3\sqrt{W_\alpha}}}$$
$$\beta = \sqrt{\frac{2 - 2\sqrt{W_\alpha}}{3W_\beta}}$$

Figure G.1 Two-Dimensional Eight-Point Integration Rule

The solution of these equations produces the following locations of the eight points and their weighting factors:

$$\alpha = \sqrt{\frac{7}{9}} \qquad \beta = \sqrt{\frac{7}{15}} \qquad w_\alpha = \frac{9}{49} \qquad w_\beta = \frac{40}{49} \tag{G.20}$$

It is apparent that the eight-point two-dimensional rule has the same accuracy as the 3 by 3 Gauss rule. Note that the sum of the eight weighting factors is 4.0, the area of the element.

G.5 AN EIGHT-POINT LOWER ORDER RULE

A lower order, or reduced, integration rule can be produced by not satisfying the equation associated with a_{40} in Equation G.19. This allows the weighting factor w_β to be arbitrarily specified. Or:

$$w_\beta = ? \quad w_\alpha = 1.0 - w_\beta \quad \alpha = \frac{1}{3\sqrt{w_\alpha}} \quad \beta = \sqrt{\frac{2 - 2\sqrt{w_\alpha}}{3w_\alpha}} \tag{G.21}$$

Therefore, if $w_\beta = 0$ the rule reduces to the 2 by 2 Gauss rule. If w_β is set to 40/49, the accuracy is the same as the 3 by 3 Gauss rule.

G.6 A FIVE-POINT INTEGRATION RULE

Using the same approach, a five-point integration rule, shown in Figure G.2, can be produced.

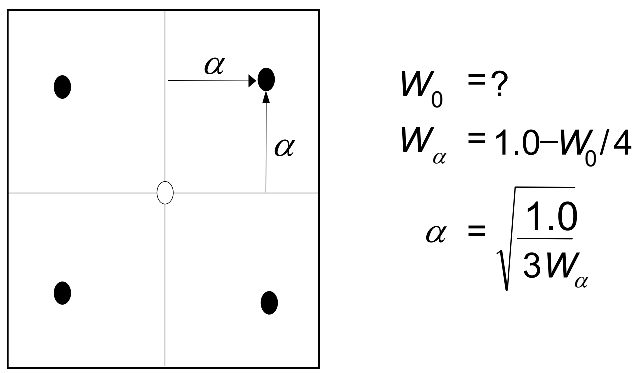


Figure G.2 Five-Point Integration Rule

The two-dimensional five-point rule can be written as:

$$I = w_\alpha \, f(\pm\alpha, \pm\alpha) + w_0 \, f(0,0) \tag{G.22}$$

Equating all non-zero terms in the integrated polynomial of the third order produces the following two equations in terms of three unknowns:

$$\begin{aligned} a_{00} &: \quad 4w_\alpha + 4w_\beta = 4 \\ a_{20}\; a_{02} &: \quad 4w_\alpha \alpha^2 = 4/\;3 \end{aligned} \tag{G.23}$$

This has the same, or greater, accuracy as the 2 by 2 Gauss rule for any value of the center node weighting value. The two-dimensional five-point numerical integration rule is summarized as:

$$w_0 = ? \quad w_\alpha = (4 - w_0)/\;4 \quad \text{and} \quad \alpha = \sqrt{\frac{1}{3w_\alpha}} \tag{G.24}$$

This equation is often used to add stability to an element that has rank deficiency when 2 by 2 integration is used. For example, the following rule has been used for this purpose:

$$w_0 = .004 \quad w_\alpha = 0.999 \quad \text{and} \quad \alpha = 0.5776391 \tag{G.25}$$

Because the five-point integration rule has a minimum of third order accuracy for any value of the center weighting value, the following rule is possible:

$$w_0 = 8/\,3 \quad w_\alpha = 1/\,3 \quad \text{and} \quad \alpha = 1.00 \tag{G.26}$$

Therefore, the integration points are at the center node and at the four node points of the two-dimensional element. Hence, for this rule it is not necessary to project integration point stresses to estimate node point stresses.

G.7 THREE-DIMENSIONAL INTEGRATION RULES

The one dimensional Gauss rules can be directly extended to numerical integration within three dimensional elements in the r, s and t reference system. However, the 3 by 3 by 3 rule requires 27 integration points and the 2 by 2 by 2 rule requires 8 points. In addition, one cannot derive the benefits of reduced integration from the direct application of the Gauss rules. Similar to the case of two-dimensional elements, one can produce more accurate and useful elements by using fewer points.

{ XE "Numerical Integration Rules:14 Point 3D Rule" }First, consider a three-dimensional, 14-point, numerical integration rule that is written in the following form:

$$I = w_\alpha\, f(\pm\alpha,\pm\alpha,\pm\alpha) + w_\beta\,[f(\pm\beta,0,0) + f(0,\pm\beta,0) + f(0,0,\pm\beta)] \tag{G.27}$$

A general, three-dimensional polynomial is of the following form:

$$f(r,s,t) = \sum_{n,m,l} a_{nml} r^n s^m t^l \tag{G.28}$$

A typical term in Equation (G.27) may be integrated exactly. Or:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} a_{nml} r^n s^m t^l \, dr\, ds\, dt = \frac{8\, a_{nml}}{(n+1)(m+1)(l+1)} \tag{G.29}$$

If n, m and l are all even numbers, Equation (G.29) is non-zero; however, for all other cases, the integral is zero. As in the case of two dimensions, equating all non-zero terms of the fifth order produces the following set of four equations in terms of four unknowns:

$$\begin{aligned}
a_{000}: &\quad 8w_\alpha + 6w_\beta = 8 \\
a_{200}\quad a_{020}\ a_{002}: &\quad 8w_\alpha \alpha^2 + 2w_\beta \beta^2 = 8/\ 3 \\
a_{220}\quad a_{022}\ a_{202}: &\quad 8w_\alpha \alpha^4 = 8/\ 9 \\
a_{400}\quad a_{040}\ a_{004}: &\quad 8w_\alpha \alpha^4 + 2w_\beta \beta^4 = 8/\ 5
\end{aligned} \tag{G.30}$$

The exact solution of these equations produces the following locations and numerical weighting values:

$$\alpha = \sqrt{\frac{19}{33}} \qquad \beta = \sqrt{\frac{19}{30}} \qquad w_\alpha = \frac{121}{361} \qquad w_\beta = \frac{320}{361} \tag{G.31}$$

Note that the sum of the weighting values is equal to 8.0, the volume of the element.

A nine-point numerical integration rule, with a center point, can be derived that has the following form:

$$I = w_\alpha\, f(\pm\alpha,\pm\alpha,\pm\alpha) + w_0\, f(0,0,0) \tag{G.32}$$

The nine-point rule requires that the following equations be satisfied:

$$\begin{aligned}
a_{000}: &\quad 8w_\alpha + w_0 = 8 \\
a_{200}\quad a_{020}\ a_{002}: &\quad 8w_\alpha \alpha^2 = 8/\ 3
\end{aligned} \tag{G.33}$$

This is a third order rule, where the weight at the center point is arbitrary, that can be summarized as

$$w_0 = ? \qquad w_\alpha = 1.0 - w_0 / 8 \qquad \alpha = \sqrt{\frac{1}{3w_\alpha}} \tag{G.34}$$

A small value of the center point weighting function can be selected when the standard 2 by 2 by 2 integration rule produces a rank deficient stiffness matrix.

In addition, the following nine-point three-dimensional rule is possible:

$$w_0 = 16/3 \qquad w_\alpha = 1/3 \qquad \alpha = 1.0 \tag{G.35}$$

For this third order accuracy rule, the eight integration points are located at the eight nodes of the element.

A six-point three-dimensional integration rule can be developed that has the six integration points at the center of each face of the hexahedral element. The form of this rule is:

$$I = w_\alpha [f(\pm\beta,0,0) + f(0,\pm\beta,0) + f(0,0,\pm\beta)] \tag{G.36}$$

Equating all non-zero terms up to the third order produces the following two equations:

$$\begin{aligned} a_{000} :& \quad 6w_\beta = 8 \\ a_{200} \quad a_{020} \quad a_{002} :& \quad 2w_\beta\beta^2 = 8/3 \end{aligned} \tag{G.37}$$

{ XE "Numerical Integration Rules:6 Point 3D Rule" }Therefore, the location of the integration points and weighting values for the six point rule is:

$$\beta = 1.0 \qquad w_\beta = 4/3 \tag{G.38}$$

The author has had no experience with this rule. However, it appears to have some problems in the subsequent calculation of node point stresses.

G.8 SELECTIVE INTEGRATION

{ XE "Selective Integration" }One of the first uses of ***selective integration*** was to solve the problem of shear locking in the four-node plane element. To eliminate the shear locking, a one-point integration rule was used to integrate the shear energy only. A 2 by 2 integration rule was used for the normal stress. This selected integration approach produced significantly improved results. Since the introduction of corrected incompatible elements, however, selective integration is no longer used to solve this problem.

For many coupled field problems, which involve both displacements and pressure as unknowns, the use of different order integration on the pressure and displacement field may be required to obtain accurate results. In addition, for fluid-like elements, a different order integration of the volume change function has produced more accurate results than the use of the same order of integration for all variables.

G.9 SUMMARY

In this appendix, the fundamentals of numerical integration in one, two and three dimensions are presented. By using the principles presented in this appendix, many different rules can be easily derived

The selection of a specific integration method requires experimentation and a physical understanding of the approximation used in the formulation of the finite element model. The use of reduced integration (lower order) and selective integration has proven to be effective for many problems. Therefore, one should not automatically select the most accurate rule. Table G.1 presents a summary of the rules derived in this appendix.

Table G.1 Summary of Numerical Integration Rules

RULE	Number of Points	Location of Points			Weighting Values		
		α	β	0	w_α	w_β	w_0
One Dimensional-Gauss $I = \int_{-1}^{1} f(r)\, dr$	1	-	-	0	-	-	2
	2	$\pm \frac{1}{\sqrt{3}}$	-	-	1.0	-	-
	3	$\pm \sqrt{\frac{3}{5}}$	-	0	$\frac{5}{9}$	-	$\frac{8}{9}$
Two Dimensional $I = \int_{-1}^{1}\int_{-1}^{1} f(r,s)\, dr\, ds$	5	$\pm \frac{1}{\sqrt{3w_\alpha}}$	-	0	$w_\alpha = 1 - w_0 / 4$	-	$w_0 = ?$
	5	± 1	-	0	$\frac{1}{3}$	-	$\frac{8}{3}$
	8	$\pm \sqrt{\frac{7}{9}}$	$\pm \sqrt{\frac{7}{15}}$	-	$\frac{9}{49}$	$\frac{40}{49}$	-
Three Dimensional $I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(r,s,t)\, dr ds dt$	9	$\pm \frac{1}{\sqrt{3w_\alpha}}$	-	0	$w_\alpha = 1 - w_{0/8}$	-	$w_0 = ?$
	14	$\pm \sqrt{\frac{19}{33}}$	$\pm \sqrt{\frac{19}{30}}$	-	$\frac{121}{361}$	$\frac{320}{361}$	-