APPENDIX D

THE EIGENVALUE PROBLEM

Eigenvalues and Eigenvectors are Properties of the Equations that Simulate the Behavior of a Real Structure

D.1 INTRODUCTION

The classical mathematical eigenvalue problem is defined as the solution of the following equation:

$$\mathbf{A}\mathbf{v}_n = \lambda_n \mathbf{v}_n \quad n = 1.....N \tag{D.1}$$

The N by N $\mathbf{A}$ matrix is real and symmetric; however, it may be singular and have zero eigenvalues λ_n. A typical eigenvector $\mathbf{v}_n$ has the following orthogonality properties:

$$\begin{aligned} &\mathbf{v}_n^T \mathbf{v}_n = 1 \quad \text{and} \quad \mathbf{v}_n^T \mathbf{v}_m = 0 \quad \text{if } n \neq m, \text{ therefore} \\ &\mathbf{v}_n^T \mathbf{A} \mathbf{v}_n = \lambda_n \text{ and } \mathbf{v}_n^T \mathbf{A} \mathbf{v}_m = 0 \quad \text{if } n \neq m \end{aligned} \tag{D.2}$$

If all eigenvectors $\mathbf{V}$ are considered, the problem can be written as:

$$\mathbf{A}\mathbf{V} = \Omega\mathbf{V}\Omega \quad \text{or} \quad \mathbf{V}^{\mathbf{T}}\mathbf{A}\mathbf{V} = \Omega\Omega \tag{D.3}$$

There are many different numerical methods to solve Equation (D.3) for eigenvectors $\mathbf{V}$ and the diagonal matrix of eigenvalues Ω. In structural analysis, in general, it is only necessary to solve for the exact eigenvalues of small matrices (if subspace iteration method is used). Therefore, the most reliable and robust will be selected because the computational time will always be relatively small. For the determination of the dynamic mode shapes and frequencies of

large structural systems, subspace iteration or Load Dependent Ritz, LDR, vectors are the most efficient approaches.

D.2 THE JACOBI METHOD

One of the oldest and most general approaches for the solution of the classical eigenvalue problem is the Jacobi method that was first presented in 1846. This is a simple iterative algorithm in which the eigenvectors are calculated from the following series of matrix multiplications:

$$\mathbf{V} = \mathbf{T}^{(0)}\mathbf{T}^{(1)}\mathbf{.....T}^{(k)}\mathbf{.....T}^{(n-1)}\mathbf{T}^{(n)} \tag{D.4}$$

The starting transformation matrix $\mathbf{T}^{(0)}$ is set to a unit matrix. The iterative orthogonal transformation matrix $\mathbf{T}^{(k)}$, with four non-zero terms in the i and j rows and columns, is of the following orthogonal form:

$$\mathbf{T}^{(k)} = \begin{bmatrix} - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & T_{ii} & - & - & T_{ij} & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & T_{ji} & - & - & T_{jj} & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \end{bmatrix} \tag{D.5}$$

The four non-zero terms are functions of an unknown rotation angle θ and are defined by:

$$T_{ii} = T_{jj} = \cos\theta \quad \text{and} \quad T_{ji} = -T_{ij} = \sin\theta \tag{D.6}$$

Therefore, $\mathbf{T}^{(k)T}\mathbf{T}^{(k)} = \mathbf{I}$, which is independent of the angle θ. The typical iteration involves the following matrix operation:

$$\mathbf{A}^{(k)} = \mathbf{T}^{(k)T}\mathbf{A}^{(k-1)}\mathbf{T}^{(k)} \tag{D.7}$$

The angle is selected to force the terms i,j and j,i in the matrix $\mathbf{A}^{(k)}$ to be zero. This is satisfied if the angle is calculated from:

$$\tan 2\theta = \frac{2A_{ij}^{(k-1)}}{A_{ii}^{(k-1)} - A_{jj}^{(k-1)}} \tag{D.8}$$

The classical Jacobi eigenvalue algorithm is summarized within the computer subroutine given in Table D.1.

One notes that the subroutine for the solution of the symmetric eigenvalue problem by the classical Jacobi method does not contain a division by any number. Also, it can be proved that after each iteration cycle, the absolute sum of the off-diagonal terms is always reduced. Hence, the method will always converge and yield an accurate solution for positive, zero or negative eigenvalues.

The Jacobi algorithm can be directly applied to all off-diagonal terms, in sequence, until all terms are reduced to a small number compared to the absolute value of all terms in the matrix. However, the subroutine presented uses a "threshold" approach in which it skips the relatively small off-diagonal terms and operates only on the large off-diagonal terms.

To reduce one off-diagonal term to zero requires approximately 8N numerical operations. Clearly, one cannot precisely predict the total number of numerical operation because it is an iterative method; however, experience has indicated that the total number of numerical operations to obtain convergence is the order of $10N^3$. Assuming a modern (1998) personal computer can perform over 6,000,000 operations per second, it would require approximately one second of computer time to calculate the eigenvalues and eigenvectors of a full 100 by 100 matrix.

Table D.1 Subroutine to Solve the Symmetric Eigenvalue Problem

```
      SUBROUTINE JACOBI(A,V,NEQ,NF)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(NEQ,NEQ),V(NEQ,NEQ)
!--- EIGENVALUE SOLUTION BY JACOBI METHOD ------
!     WRITTEN BY ED WILSON July 4, 2008 --------
!     A  - GIVEN MATRIX TO BE SOLVED -----------
!          EIGENVALUES ON DIAGONAL AFTER SOLUTION
!     V  - MATRIX OF EIGENVECTORS PRODUCED
!     NF - NUMBER OF SIGNIFICANT FIGURES
      DATA ZERO /0.0D0/
!--- INITALIZE EIGENVECTORS and SET TOLERANCE --
      TOL = 0.1**NF ; SUM = ZERO ; V =   ZERO
      DO I=1,NEQ
      V(I,I) = 1.0
       DO  J=1,NEQ ; SUM = SUM + ABS(A(I,J)) ; END DO
      END DO   ! "I" LOOP
      IF (SUM.LE.0.0) RETURN   ! NULL MATRIX
!--- REDUCE MATRIX TO DIAGONAL ----------------
  400 SSUM = ZERO
      DO J=2,NEQ
      IH = J - 1
       DO I=1,IH
       IF (ABS(A(I,J))/SUM.GT.TOL) THEN  ! SET A(I,J) TO ZERO
       SSUM = SSUM + ABS(A(I,J))  ! SUM OF OFF-DIAGONAL TERMS
!--- CALCULATE ROTATION ANGLE -----------------
       AA = ATAN2(2.0*A(I,J),A(I,I)-A(J,J))/2.0
       SI = SIN(AA) ;  CO = COS(AA)
!--- MODIFY "I" AND "J" COLUMNS OF "A" AND "V" -
        DO K=1,NEQ
        TT = A(K,I)
        A(K,I) =  CO*TT + SI*A(K,J)
        A(K,J) = -SI*TT + CO*A(K,J)
        TT = V(K,I)
        V(K,I) =  CO*TT + SI*V(K,J)
        V(K,J) = -SI*TT + CO*V(K,J)
        END DO
!--- MODIFY DIAGONAL TERMS --------------------
       A(I,I) = CO*A(I,I) + SI*A(J,I)
       A(J,J) =-SI*A(I,J) + CO*A(J,J)
       A(I,J) = ZERO
!--- MAKE "A" MATRIX SYMMETRICAL --------------
       DO K=1,NEQ ; A(I,K)=A(K,I) ; A(J,K)=A(K,J) ; END DO
       END IF
       END DO  ! "J" LOOP
      END DO   ! "I" LOOP
!--- CHECK FOR CONVERGENCE --------------------
      IF( ABS(SSUM)/SUM .GT.TOL ) GO TO 400
      RETURN ; END
```

D.3 CALCULATION OF 3D PRINCIPAL STRESSES

The calculation of the principal stresses for a three-dimensional solid can be numerically evaluated from the stresses in the x-y-z system by solving a cubic equation. However, the definition of the directions of the principal stresses is not a simple procedure. An alternative approach to this problem is to write the basic stress transformation equation in terms of the unknown directions of the principal stresses in the 1-2-3 reference system. Or:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} V_{x1} & V_{y1} & V_{z1} \\ V_{x2} & V_{y2} & V_{z2} \\ V_{x3} & V_{y3} & V_{z3} \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} V_{x1} & V_{x2} & V_{x3} \\ V_{y1} & V_{y2} & V_{y3} \\ V_{z1} & V_{z2} & V_{z3} \end{bmatrix} \tag{D.9}$$

Or, in symbolic form:

$$\Omega\Omega = \mathbf{V}^{\mathrm{T}}\,\mathbf{S}\,\mathbf{V} \tag{D.10}$$

in which $\mathbf{V}$ is the standard direction cosine matrix. Because $\mathbf{V}\,\mathbf{V}^{\mathrm{T}}$ is a unit matrix, Equation (D.3) can be written as the following eigenvalue problem:

$$\mathbf{SV} = \mathbf{V}\Omega \tag{D.11}$$

where Ω is an unknown diagonal matrix of the principal stresses (eigenvalues) and $\mathbf{V}$ is the unknown direction cosine matrix (eigenvectors) that uniquely define the directions of the principal stresses. To illustrate the practical application of the classical Jacobi method, consider the following state of stress:

$$\mathbf{S} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 120 & -55 & -75 \\ -55 & -55 & 33 \\ -75 & 33 & -85 \end{bmatrix} \tag{D.12}$$

The eigenvalues, principal stresses, and eigenvectors (direction cosines) are:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} = \begin{bmatrix}162.54\\ -68.40\\ -114.14\end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix}.224 & .352 & .909\\ -.308 & .910 & -.277\\ .925 & .217 & -.312\end{bmatrix} \tag{D.13}$$

The solution of a 3 by 3 eigenvalue problem can be considered as a trivial numerical problem. Several hundred of those problems can be solved by the classical Jacobi method in one second of computer time. Note that negative eigenvalues are possible.

D.4 SOLUTION OF THE GENERAL EIGENVALUE PROBLEM

The general eigenvalue problem is written as:

$$\mathbf{AV} = \Omega\mathbf{BV}\Omega \tag{D.14}$$

where both **A** and **B** are symmetrical matrices. The first step is to calculate the eigenvectors $\mathbf{V}_B$ of the **B** matrix. We can now let the eigenvectors **V** be a linear combination or the eigenvectors of the **B** matrix. Or:

$$\mathbf{V} = \Omega\mathbf{V}_B\overline{\mathbf{V}} \tag{D.15}$$

Substitution of Equation (D.15) into Equation (D.14) and the pre-multiplication of both sides by $\mathbf{V}_B^T$ yields:

$$\mathbf{V}_B^T\mathbf{A}\mathbf{V}_B\overline{\mathbf{V}} = \Omega\mathbf{V}_\mathrm{B}^\mathrm{T}\mathbf{B}\mathbf{V}_\mathrm{B}\overline{\mathbf{V}}\,\Omega \tag{D.16}$$

If all eigenvalues of the **B** matrix are non-zero, the eigenvectors can be normalized so that $\mathbf{V}_B^T\mathbf{B}\mathbf{V}_B = \mathbf{I}$. Hence, Equation (D.16) can be written in the following classical form:

$$\overline{\mathbf{A}}\,\overline{\mathbf{V}} = \overline{\mathbf{V}}\Omega \tag{D.17}$$

where $\overline{\mathbf{A}} = \mathbf{V}_B^T\mathbf{A}\mathbf{V}_B$. Therefore, the general eigenvalue problem can be solved by applying the Jacobi algorithm to both matrices. If the **B** matrix is diagonal, the eigenvectors $\mathbf{V}_B$ matrix will be diagonal, with the diagonal terms equal to $1/\sqrt{B_{nn}}$. This is the case for a lumped mass matrix. Also, mass must be

associated with all degrees of freedom and all eigenvectors and values must be calculated.

D.5 SUMMARY

Only the Jacobi method has been presented in detail in this section. Traditionally, it has been restricted to small full matrices in which all eigenvalues are required. For the dynamic modal analysis of very large structural systems, where all the modes are not required, the basic Jacobi method, coupled with LDR vectors, produces a very robust method. This new method produces static modes, rigid-body modes and the classical dynamic modes. These three types of modes has allowed the Fast Nonlinear Analysis, FNA, method to be the fastest and most accurate solution approach for a large class of nonlinear dynamic structural problems. This new Jacobi method has been used in SAP2000 for the past ten years with impressive results.

Carl Gustav Jacob Jacobi (10 December 1804 –February 1851) was a German mathematician, who made fundamental contributions to classical mechanics, dynamics and astronomy. A crater on the Moon is named after him. He earned the degree of Doctor of Philosophy in 1825 at Berlin University. He was a great teacher and researcher at the University of Konigsberg until 1842.

One of his maxims was: 'Invert, always invert', expressing his belief that the solution of many hard problems can be clarified by re-expressing them in inverse form.

He was one of the early founders of the theory of determinants; in particular, he invented the Jacobian determinant formed from the n^2 differential coefficients of n given functions of n independent variables, and which has played an important part in many analytical investigations. In 1841 he reintroduced the partial derivative ∂ notation of Legendre, which was to become standard. These are mathematical notations that engineer still use in the development of finite elements. Jacobi first presented the method for the determination of eigenvalues and eigenvectors in 1846.